Math 324 D | Quiz 4 (20 minutes) | August 12, 2015

Name: Key

Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. Leave your answers in *exact form* instead of decimal approximations.

1. Suppose a metal ball of radius r and conductivity k is centered about a heat source at the origen. Suppose the temperature at a point in the ball is inversely proportional to 1 plus the square of its distance from the heat source, i.e., if d is the distance from the origen,

$$T = \frac{C}{1+d^2}.$$

We will compute the rate of heat flow across the surface of the metal ball.

(a) (2 points) Recall that heat flow of a temperature function is the vector field $\vec{F} = -k\nabla T$. Compute the heat flow and its curl.

$T(x,y,z) = \frac{C}{1+x^2+y^2+z^2}$

so $\nabla T(x,y,z) = \frac{C\langle -2x, -2y, -2z\rangle}{(1+x^2+y^2+z^2)^2}$

Thus $\vec{F} = -k\nabla T = \frac{2kC}{(1+x^2+y^2+z^2)^2}\langle x, y, z\rangle$

$\vec{F}$ conservative $\Rightarrow \nabla \vec{F} = 0$.

(b) (3 points) What is the unit normal vector $\vec{n}$ on the surface of the metal ball? What is $\vec{F}\cdot\vec{n}$? Can you say anything more about the value of $\vec{F}\cdot\vec{n}$ on the surface of the ball?

Unit normal is in radial direction.

$\vec{n} = \frac{\langle x, y, z\rangle}{r}$

$\vec{F}\cdot\vec{n} = \frac{2kC}{r(1+x^2+y^2+z^2)^2}(x^2+y^2+z^2)$

on S, $x^2+y^2+z^2 = r^2$ so

$\vec{F}\cdot\vec{n} = \frac{2kCr}{(1+r^2)^2}$

(c) (3 points) Recall that the rate of heat flow through a surface S is the flux of $\vec{F}$ through S. Set up and integral for the rate of heat flow across the surface of the metal ball.

Rate Heat Flow $= \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \frac{2kCr}{(1+r^2)^2} dS$

(d) (2 points) Use the fact that the surface area of the sphere is $4\pi r^2$ to compute the rate of heat flow across the surface of the metal ball.

$\frac{2kCr}{(1+r^2)^2}$ is constant. So from (c)

Rate of Heat Flow $= \iint_S \frac{2kCr}{(1+r^2)^2} dS = \frac{2kCr}{(1+r^2)^2}\iint_S dS$

$= \frac{2kCr}{(1+r^2)^2}\text{Area}(S) = \boxed{\frac{8\pi kCr^3}{(1+r^2)^2}}$