Key

Midterm

Math 324 D Summer 2015

Name: ______________________________

Directions:

- You have 60 minutes to complete this exam.
- Give all answers in exact form unles stated otherwise.
- Only non-graphing calculators are allowed.
- You are allowed one hand-written sheet of notes on regular 8.5-11 paper. You may use both sides
- You must show your work.
- Circle or box your final answers.
- If you run out of space, use the back page and indicate that you have done so.
- If you have any questions, raise your hand. GOOD LUCK!

Question	Points	Score
1	20	
2	15	
3	10	
4	10	
5	10	
Total:	65	

1. (a) (4 points) Let T be a transformation defined by the equations $x = 8u^2 + uv$ and $y = uv^3$. Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of this transformation in terms of u and v.

$$\frac{\partial x}{\partial u} = 16u + v \qquad \frac{\partial y}{\partial u} = v^3$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 3uv^2$$

+2

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 16u+v & v^3 \\ u & 3uv^2 \end{vmatrix} = 48u^2v^2 + 3uv^3 - uv^3$$

+1

$$= 48u^2v^2 + 2uv^3$$

+1

(b) (4 points) Let $F(x, y, z) = \sin xy + xe^{3yz}$. Compute the gradient $\nabla F(x, y, z)$.

$$F_x = y\cos xy + e^{3yz}$$

$$F_y = x\cos xy + 3xze^{3yz}$$

+2

$$F_z = 3xye^{3yz}$$

$$\nabla F = \langle y\cos xy + e^{3yz}, x\cos xy + 3xze^{3yz}, 3xye^{3yz} \rangle$$

+2

(c) (4 points) Let $g(x, y) = x^3 + xy + 4y^3$. The equation $g(x, y) = 0$ implictly defines y as a function of x. Use implicit differentiation to find $\frac{dy}{dx}|_{(1,1)}$.

$$\frac{dy}{dx} = \frac{-g_x(x,y)}{g_y(x,y)}$$

$$= -\frac{3x^2+y}{x+12y^2} \quad +2$$

$$+1 \quad g_x = 3x^2 + y$$

$$g_y = x + 12y^2$$

$$\frac{dy}{dx}\Big|_{(1,1)} = \frac{-4}{13} \quad +1$$

(d) (4 points) Consider $z = g(x, y)$ with $g(x, y)$ as in part (c). Let $\vec{u}$ be the vector forming an angle of $\pi/3$ with the positive x-axis (i.e., in the direction of the first quadrant). Compute the directional derivative $D_{\vec{u}}g(1, 1)$.

$$\vec{u} = \langle \cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3} \rangle = \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle \quad +1$$

$$D_{\vec{u}}g(x,y) = g_x \cdot \tfrac{1}{2} + g_y \cdot \tfrac{\sqrt{3}}{2} = \tfrac{1}{2}(3x^2+y) + \tfrac{\sqrt{3}}{2}(x+12y^2) \quad +2$$

$$D_{\vec{u}}g(1,1) = 2 + \frac{13\sqrt{3}}{2} \quad +1$$

(e) (4 points) Consider $z = g(x, y)$ as in part (c). Suppose further that $x = \sin t$ and $y = \cos t$. Use the chain rule to compute $\frac{dz}{dt}$ in terms of t, and evaluate it at $t = 0$.

$$\frac{dg}{dt} = \frac{dg}{dx}\cdot\frac{dx}{dt} + \frac{dg}{dy}\cdot\frac{dy}{dt} \quad +1$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dt} = -\sin t \quad +1$$

$$= (3\sin^2 t + \cos t)(\cos t) + (\sin t + 12\cos^2 t)(-\sin t) \quad +1$$

@ $t = 0$

$$= (0+1)(1) + (0+12)(0) = 1 \quad +1$$

2. (15 points) Let H be the upper hemisphere of radius 1. Suppose $\rho(x, y, z) = z^2 + 1$ is a density function of H. Compute the mass and center of mass of H. *Hint*: You can argue by symmetry that 2 of the moments are 0.

$$\text{Mass} = \iiint_H (z^2+1)\,dV.$$

All symmetry about z-axis $\Rightarrow$ try cylindrical.

$$+3 \quad = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} (z^2+1)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{(1-r^2)^{3/2}}{3} + \sqrt{1-r^2}\right] r\,dr\,d\theta$$

Let $u = 1-r^2$

so $du = -2r\,dr$

$$= -\frac{1}{2}\int_0^{2\pi}\int_1^0 \left(\frac{u^{3/2}}{3} + \sqrt{u}\right) du\,d\theta$$

$$= -\pi\left[\frac{1}{3}\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_1^0$$

$$= -\pi\left[0 - \left[\frac{2}{15} + \frac{2}{3}\right]\right]$$

$$+2 \quad = \boxed{\frac{12\pi}{15}} = \frac{4\pi}{5}$$

Because H and ρ are symmetric about the z-axis, the center of mass must lie on it, so (+3)

$$\bar{x} = \bar{y} = 0.$$

$$\bar{z} = \frac{M_{xy}}{m} \quad (+1)$$

$$M_{xy} = \iiint_H z(z^2+1)\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} (z^3+z)\,r\,dz\,dr\,d\theta \quad (+3)$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{(1-r^2)^2}{4} + \frac{1-r^2}{2}\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{3r}{4} - r^3 + \frac{r^5}{4}\right) dr\,d\theta$$

$$= 2\pi\left[\frac{3}{8} - \frac{1}{4} + \frac{1}{24}\right]$$

$$= \pi/3$$

So $$\bar{z} = \frac{\pi/3}{4\pi/5} = \frac{5}{12} \quad (+2)$$

So the center of mass is

$$\left(0, 0, \tfrac{5}{12}\right) \quad (+1)$$

3. (a) (4 points) 2 boats travelling on a lake produce orthogonal sine waves on the surface of the water. As a result, the surface of the water is given by the equation

$$f(x) = \sin x + \sin y.$$

Set up **BUT DO NOT COMPUTE** the integral for the surface area of the water on the unit square: $[0,1] \times [0,1]$.

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA \qquad f_x = \cos x \quad f_y = \cos y$$

$$= \int_0^1 \int_0^1 \sqrt{\cos^2 x + \cos^2 y + 1}\, dA$$

(b) (6 points) Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx,$$

so that it ends with $dxdydz$. **AGAIN, DO NOT COMPUTE.**

step 1 1 by 1.

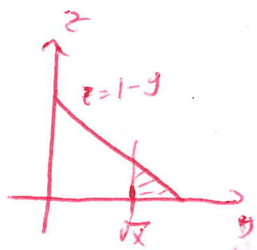
Pass z by y
x const

+2

$$\int_{\sqrt{x}}^1 \int_0^{1-y} f\,dz\,dy$$

$$\parallel$$

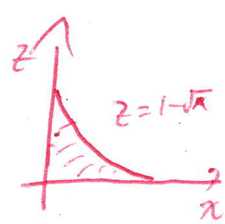
$$\int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f\,dy\,dz$$



step 2 Pass x by z

$$\int_0^1 \int_0^{1-\sqrt{x}} g\,dz\,dx = \int_0^1 \int_0^{(1-z)^2} g\,dx\,dz$$

+2

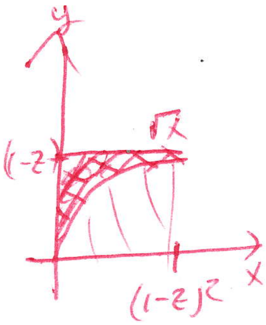


step 3 pass x by y (z const)

$$\int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\,dy\,dx$$

$$\parallel$$

$$\int_0^{1-z} \int_0^{y^2} f\,dx\,dy$$



So

$$\boxed{\int_0^1 \int_0^{1-z} \int_0^{y^2} f\,dx\,dy\,dz}$$

4. (10 points) Compute the volume of the solid enclosed by the 3 coordinate planes and the surface $x + 2y + 5z = 10$.

$$E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} 0 \le z \le 2 \\ 0 \le y \le 5 - \frac{5z}{2} \\ 0 \le x \le 10 - 5z - 2y \end{array} \right\} \quad +3$$

$$\text{Vol} = \iiint_E dV = \int_0^2 \int_0^{5-5/2 z} \int_0^{10-5z-2y} dx\,dy\,dz \quad +3$$

+2

$$= \int_0^2 \int_0^{5-\frac{5z}{2}} (10 - 5z - 2y)\,dy\,dz$$

$$= \int_0^2 10\left[5-\frac{5z}{2}\right] - 5z\left[5-\frac{5z}{2}\right] - \left[5-\frac{5z}{2}\right]^2 dz$$

$$= \int_0^2 \left(50 - 25z - 25z + \frac{25z^2}{2} - 25 + 25z - \frac{25z^2}{4}\right) dz$$

$$= \int_0^2 \left(25 - 25z + \frac{25z^2}{4}\right) dz$$

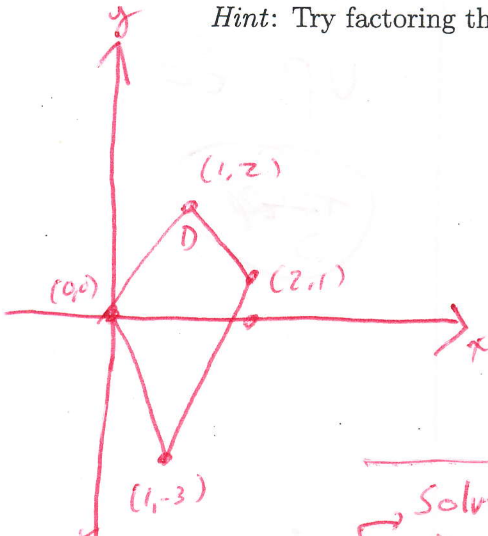
$$= 25(2) - \frac{25(2)^2}{2} + \frac{25}{4} \cdot \frac{(2)^3}{3}$$

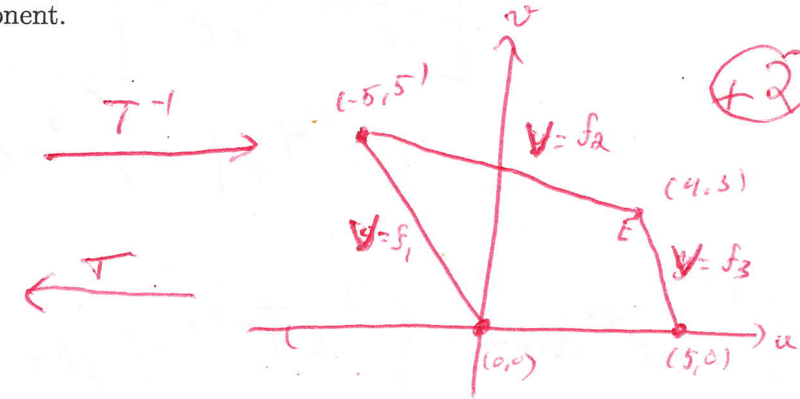
$$= 50 - 50 + \frac{200}{12} = \boxed{\frac{50}{3}} \quad +2$$

5. (10 points) Let D be the parallelogram with vertices $(0,0), (1,2), (2,1)$, and $(1,-3)$. Compute the following integral by making an appropriate change of variables.

$$\iint_D (x+2y)e^{2x^2+3xy-2y^2}\,dA.$$

Hint: Try factoring the exponent.





+2

Let $u = x+2y$, $v = 2x-y$.

Solve $x = \frac{u}{5} + \frac{2v}{5}$ +2

$y = \frac{2u}{5} - \frac{v}{5}$

So $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{vmatrix} = \frac{-1}{5}$ +2

Then we have

(x,y)	(u,v)
$(0,0)$	$(0,0)$
$(1,2)$	$(5,0)$
$(2,1)$	$(4,3)$
$(1,-3)$	$(-5,5)$

f_1 is $-u$

f_2 is $\frac{-2}{9}u + \frac{35}{9}$

f_3 is $-3u + 15$

$E = E_1 \cup E_2 \cup E_3$

w/ $E_1 = \{-5 \le u \le 0,\ f_1 \le v \le f_2\}$

$E_2 = \{0 \le u \le 4,\ 0 \le v \le f_2\}$

$E_3 = \{4 \le u \le 5,\ 0 \le v \le f_3\}$

So

$$\iint_D (x+2y)e^{2x^2+3xy-2y^2}\,dA$$

$$= \frac{1}{5}\iint_E u e^{uv}\,dA \quad +4$$

$$= \frac{1}{5}\left[\int_{-5}^{0}\int_{-u}^{-\frac{2}{9}u+\frac{35}{9}} ue^{uv}\,dv\,du + \int_0^4\int_0^{-\frac{2}{9}u+\frac{35}{9}} ue^{uv}\,dv\,du + \int_4^5\int_0^{-3u+15} ue^{uv}\,dv\,du\right]$$

1 2 3

Full credit here

Extra credit

① $$\int_{-5}^{0}\int_{-u}^{-\frac{2}{9}u+5} ue^{uv}\,dv\,du = \int_{-5}^{0}\left[e^{-\frac{2}{9}u+5} - e^{-u}\right]du$$

$$= \left[-\frac{9}{2}e^{-2/9u+5} + e^{u}\right]_{-5}^{0}$$

$$= -\frac{9}{2}e^{35/9} + 1 + \frac{9}{2}e^{5} - e^{-5}$$

② $$\int_{0}^{4}\int_{0}^{-\frac{2}{9}u+5} ue^{uv}\,dv\,du = \int_{0}^{4}\left(e^{-\frac{2}{9}u+5} - 1\right)du$$

$$= -\frac{9}{2}e^{3} - 4 + \frac{9}{2}e^{35/9}$$

③ $$\int_{4}^{5}\int_{0}^{-3u+15} ue^{uv}\,dv\,du = \int_{4}^{5}\left(e^{-3u+15} - 1\right)du$$

$$= \left[-\frac{1}{3}e^{-3u+15} - u\right]_{4}^{5}$$

$$= -\frac{1}{3} - 5 + \frac{1}{3}e^{3} - 4$$

So in total

$$\boxed{\frac{1}{5}\left[\frac{9}{2}e^{5} - \frac{25}{6}e^{3} - \frac{37}{3} - e^{-5}\right]}$$

up to

+5