Final; Part 2

Math 324 D Summer 2015

Name: ____

Directions:

- You have 60 minutes to complete this exam.
- Give all answers in exact form unless stated otherwise.
- Only non-graphing calculators are allowed.
- You are allowed one hand-written sheet of notes on regular 8.5-11 paper. You may use both sides
- You must show your work.
- Circle or box your final answers.
- If you run out of space, use the back page and indicate that you have done so.
- If you have any questions, raise your hand. GOOD LUCK!

Question	Points	Score
1	12	
2	8	
3	10	
4	4	
5	6	
Total:	40	

1. Let $y = \sqrt{x^2+z^2}$ with $1 \le y \le 4$ be a section of a cone with density $\rho(x,y,z) = y+1$.

(a) (4 points) Parametrize the cone as a vector valued function $\vec{r}(u,v)$, and compute the correction factor, $|r_u \times r_v|$.

$\vec{r}(u,v) = u\vec{i} + \sqrt{u^2+v^2}\vec{j} + v\vec{k}$

(or $\vec{r}(x,z) = x\vec{i} + \sqrt{x^2+z^2}\vec{j} + z\vec{k}$) +1

since y is a function of x and z,

Note: also can compute directly.

$|r_x \times r_z| = \sqrt{\frac{\partial y}{\partial x}^2 + \frac{\partial y}{\partial z}^2 + 1} = \sqrt{\frac{x^2}{x^2+z^2} + \frac{z^2}{x^2+z^2} + 1} = \sqrt{2}$ +3

Note

can compute in terms of r & θ in part (a) as well

would get $|r_r \times r_\theta| = \sqrt{2}\, r$

(b) (4 points) Find the mass of the cone.

$\rho(x,y,z) = y+1 = \sqrt{x^2+z^2}+1$

$m = \iint_S \rho\, dS = \iint_D (\sqrt{x^2+z^2}+1)\,|r_x \times r_z|\, dA$ +1 +1

$= \iint_D (\sqrt{x^2+z^2}+1)\cdot\sqrt{2}\, dA \overset{\text{polar}}{=} \int_0^{2\pi}\int_1^4 (r+1)\sqrt{2}\, r\, dr\, d\theta$ +2

$= 2\pi\sqrt{2}\int_1^4 r^2 + r\, dr = (2\pi\sqrt{2})\left(\frac{57}{2}\right) = \boxed{57\pi\sqrt{2}}$

(c) (4 points) Find the center of mass of the cone. (Hint: use symmetry to your advantage).

All symmetric about y-axis $\Rightarrow \bar{x} = \bar{z} = 0$ +1

$\bar{y} = \frac{1}{m}\iint_S y\rho\, dS = \frac{1}{57\pi\sqrt{2}}\iint_D r\cdot(r+1)\cdot\sqrt{2}\cdot r\, dr\, d\theta$ +1 +1

$= \frac{1}{57\pi\sqrt{2}}\int_0^{2\pi}\int_1^4 \sqrt{2}(r^3+r^2)\, dr\, d\theta$

$= \frac{2}{57}\int_1^4 (r^3+r^2)\, dr$

$= \left(\frac{2}{57}\right)\left(\frac{339}{4}\right) = \frac{339}{114}$

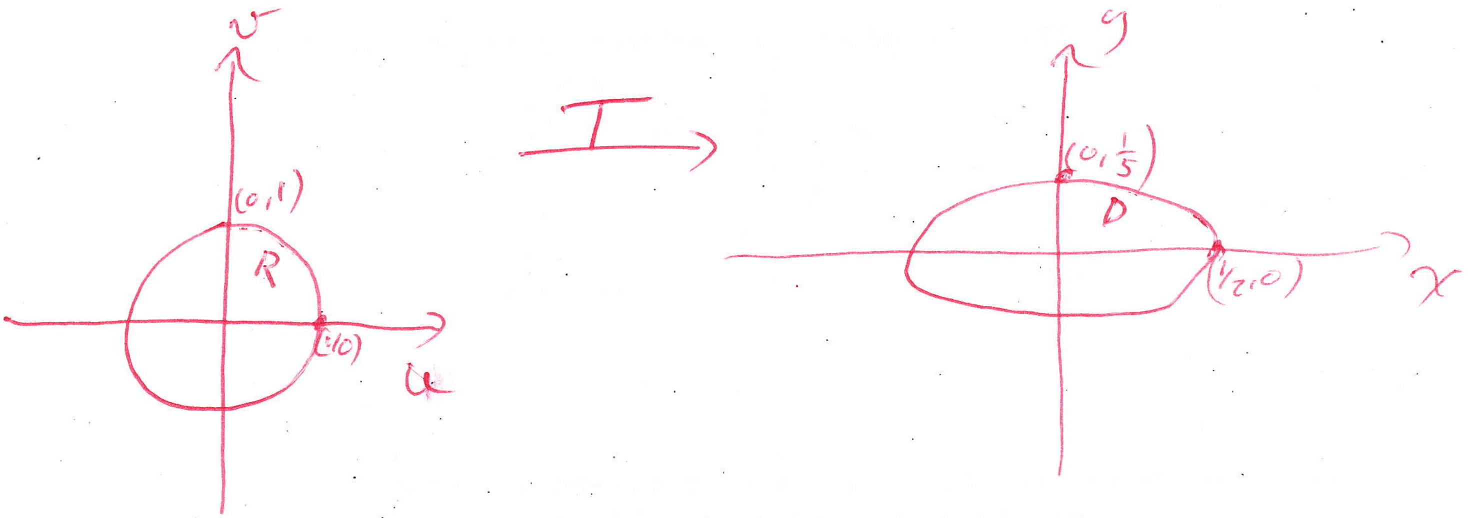
so

$\left(0, \frac{339}{114}, 0\right)$ +1

$\left(0, \frac{113}{38}, 0\right)$

2. (8 points) Let D be the region enclosed by the ellipse $4x^2 + 25y^2 = 1$. Use a suitable change of coordinates to evaluate the integral

$$\int\int_D \cos(4x^2 + 25y^2)\,dA.$$



T is the transformation $x = \frac{u}{2}$, $y = \frac{v}{5}$ mapping the unit circle R to D. +3

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/5 \end{vmatrix} = \frac{1}{10} \quad +2$$

So $\displaystyle\iint_D \cos(4x^2+25y^2)\,dA = \iint_R \cos\left(4\left(\tfrac{u}{2}\right)^2 + 25\left(\tfrac{v}{5}\right)^2\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$ +2

$$= \frac{1}{10}\iint_R \cos(u^2+v^2)\,dA = \frac{1}{10}\int_0^{2\pi}\int_0^1 \cos r^2\, r\,dr\,d\theta$$

$$= \frac{1}{20}\int_0^{2\pi}\int_0^1 \cos w\,dw\,d\theta$$

Let $w = r^2$

$dw = 2r\,dr$

$$= \frac{\pi}{10}\left[-\sin 0 + \sin 1\right] \quad +1$$

$$= \boxed{\frac{\pi}{10}\sin 1}$$

3. A fluid of conductivity $k = 1/2$ fills a cylindrical container S given by the equation $x^2 + y^2 = 9$ for $0 \le z \le 4$. The temperature is given by

$$T(x, y, z) = \frac{1}{x^2 + y^2 + 1}.$$

(a) (2 points) Compute the heat flow vector field $\vec{F} = -K\nabla T$.

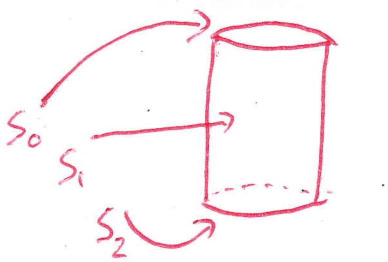
$$\frac{\partial T}{\partial x} = \frac{-2x}{(x^2+y^2+1)^2}$$

$$\frac{\partial T}{\partial y} = \frac{-2y}{(x^2+y^2+1)^2}$$

$$\frac{\partial T}{\partial z} = 0 \quad (+1)$$

$$\vec{F} = -\frac{1}{2}\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \frac{\langle x, y, 0\rangle}{(x^2+y^2+1)^2} \quad (+1)$$

(b) (8 points) The rate of heat flow accross the surface of the container is given by the flux of $\vec{F}$ through S. Compute this value. (Careful: Do not forget the top and bottom of the container).



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} \quad (+2)$$

$$\iint_{S_0} \vec{F}\cdot d\vec{S} = \iint_{S_0} \vec{F}\cdot \vec{n}\, dS$$

But $\vec{n} = \vec{k}$, and $\vec{F}\cdot\vec{k} = 0$

So $\iint_{S_0} \vec{F}\cdot d\vec{S} = 0$

Similarly $\iint_{S_2} \vec{F}\cdot d\vec{S} = 0$ because the unit normal for S_2 is $-\vec{k}$ and $\vec{F}(\vec{k}) = 0$ (+1)

Parametrize S_1:

$x = 3\cos\theta$

$y = 3\sin\theta$

$z = z$

$r_\theta = \langle -3\sin\theta, 3\cos\theta, 0\rangle$

$r_z = \langle 0, 0, 1\rangle$

outward $r_\theta \times r_z = \langle 3\cos\theta, 3\sin\theta, 0\rangle$ (+3)

In terms of θ an z

$$\vec{F} = \frac{\langle 3\cos\theta, 3\sin\theta, 0\rangle}{100}$$

note $(x^2+y^2+1)^2 = (9\cos^2\theta + 9\sin^2\theta + 1)^2 = 10^2$

So $\vec{F}\cdot(r_\theta \times r_z) = \dfrac{9\cos^2\theta + 9\sin^2\theta}{100} = \dfrac{9}{100}$

Thus $\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot(r_\theta \times r_z)\, dS$ (+2)

$$= \iint_{S_1} \frac{9}{100}\, dS = \frac{9}{100}\,\text{Area}(S_1) \to \frac{9}{100}\cdot 2\pi\cdot 3\cdot 4 = \boxed{\frac{54\pi}{25}}$$

4. (4 points) Let S, given by $\vec{r}(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + v\vec{k}$ be an infinite spiral ramp. Compute the tangent plane at (1,0,0).

$r_u = \langle \cos v, \sin v, 0 \rangle$

$r_v = \langle -u\sin v, u\cos v, 1 \rangle$

$r_u \times r_v = \langle \sin v, -\cos v, u \rangle$

this is the normal vector +2

$\vec{r}(u,v) = (1,0,0)$
$\Rightarrow u = 1$
$v = 0$

So the normal vector is $\langle \sin 0, -\cos 0, 1 \rangle = \langle 0, -1, 1 \rangle$ +2

Thus the tangent plane is

$0(x-1) - 1(y-0) + 1(z-0) = 1$ i.e. $-y + z = 0$

5. (6 points) For this question, fix a vector field $\vec{F} = \langle xy\sin z, x^2+y^2+z^2, xyz \rangle$. Also fix a function $f(x,y,z) = xyz^3$. Compute each of the following, if they make sense. If not, write "does not exist".

+1 each

(a) $\operatorname{curl}\operatorname{div}\vec{F}$

DNE

(b) $\operatorname{curl}\nabla f$

0

(c) $\operatorname{div}\nabla f$

$= \operatorname{div}\langle yz^3, xz^3, 3xyz^2 \rangle = 0 + 0 + 6xyz = \boxed{6xyz}$

(d) $\operatorname{div}\operatorname{curl}\vec{F}$.

0

(e) $\operatorname{curl} f$

DNE

(f) $\nabla \operatorname{div}\vec{F}$

$= \nabla(y\sin z + 2y + xy) = \langle y, \sin z + 2 + x, y\cos z \rangle$