Final; Part 1

Math 324 D Summer 2015

Name: KEY

Directions:

- You have 60 minutes to complete this exam.
- Give all answers in exact form unless stated otherwise.
- Only non-graphing calculators are allowed.
- You are allowed one hand-written sheet of notes on regular 8.5-11 paper. You may use both sides
- You must show your work.
- Circle or box your final answers.
- If you run out of space, use the back page and indicate that you have done so.
- If you have any questions, raise your hand. GOOD LUCK!

Question	Points	Score
1	12	
2	10	
3	10	
4	8	
Total:	40	

1. The following questions can be done quickly.

(a) (3 points) Let $\vec{F} = \langle 3x + e^{yz}, xyz^2, 2xy\sin z\rangle$ be a vector field. Is there a vector field $\vec{G}$ such that $\vec{F} = \operatorname{curl}\vec{G}$? Why or why not?

$\operatorname{div}\vec{F} = 3 + xz^2 + 2xy\cos z \neq 0$ (+2)

But

$\operatorname{div}\operatorname{curl}\vec{G} = 0$ for all G (+1)

so $\vec{F} \neq \operatorname{curl} G$ for any G.

(b) (3 points) Let $\vec{F} = \langle 3yz^2, xz + \sin z, 7y^4 + \cos x\rangle$. If S is the sphere of radius one centered at the origen, oriented outwards, determine the value of

$$\iint_S \vec{F} \cdot d\vec{S}.$$

S is the boundary of the solid ball B of radius 1. (+1)

Therefore, by the divergence theorem:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_B \operatorname{div}\vec{F}\, dV \quad (+1)$$

$$= 0$$

because $\operatorname{div}\vec{F} = 0$ (+1)

(c) (3 points) Let $f(x,y,z) = e^y \sin x + x^2 y$. Compute the gradient $\vec{F} = \nabla f(x,y,z)$.

$$\nabla f = \langle e^y \cos x + 2xy,\ e^y \sin x + x^2 \rangle.$$

+3

(d) (3 points) Let C be circle of radius 1 centered at the origen. With the same $\vec{F}$ computed in part (c), determine the value of

$$\oint_C \vec{F} \cdot d\vec{r}.$$

Explain your reasoning.

$\vec{F}$ conservative +1

so $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.

+2

2. Let $\vec{F} = \langle y, x, \cos z\rangle$ be a vector field defined on all of $\mathbb{R}^3$.

(a) (3 points) Show (without directly computing a potential) that $\vec{F}$ is conservative.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y & x & \cos z \end{vmatrix} = \langle 0, 0, 1-1\rangle = 0 \quad (+2)$$

And $\vec{F}$ is defined on all of R^3 (+1)

So $\vec{F}$ is conservative.

(b) (3 points) Compute a potential function for $\vec{F}$ (i.e., a function f so that $\nabla f = \vec{F}$).

f s.t. $\langle f_x, f_y, f_z\rangle = \vec{F}$.

$f_x = y$

$\Rightarrow f = xy + g(y,z)$ (+1)

$x = f_y = x + g_y(y,z)$

$\Rightarrow g_y(y,z) = 0$ (+1)

$\Rightarrow g(y,z) = h(z)$ some function of only z

$\cos z = f_z = h'(z)$

so (+1)

$h(z) = \sin z + K$

$f(x,y,z) = xy + \sin z$ works.

(c) (4 points) Let C be the straight line from $(0, 0, \pi/2)$ to (π, π, π). Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

By Fundamental theorem:

$$\int_C \vec{F}\cdot d\vec{r} = f(\pi,\pi,\pi) - f(0,0,\pi/2) = \pi^2 - 1 \quad (+4)$$

Or

C is given by $\vec{r}(t) = \langle \pi t, \pi t, \frac{\pi}{2} + \frac{\pi}{2}t\rangle$, (+2) $0 \le t \le 1$

Integrate $\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$ (+2)

3. Let $\vec{F} = -y\vec{i} + xz\vec{j} + e^{xyz}\vec{k}$.

(a) (3 points) Compute $\vec{G} = \operatorname{curl} \vec{F}$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ -y & xz & e^{xyz} \end{vmatrix} = \langle xze^{xyz} - x,\ -yze^{xyz},\ z+1 \rangle$$

(b) (7 points) Let C be the circle of radius 1 lying in the xy-plane oriented counte-clockwise, given by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 0\vec{k}$. Use Stokes' theorem to determine the value of

$$\int_C \vec{F} \cdot d\vec{r}.$$

(Hint: Stokes' theorem allows you to instead compute an integral of $\vec{G}$ for a suitably chosen surface whose boundary is C. Keep track of the induced orientation.)

Let D be the unit disk $\{x^2+y^2 \le 1, z=0\}$ lying on the xy-plane. (+2)
Give it the orientation $\vec{n} = \vec{k}$ (+1).
Then $\partial D = C$ w/ the correct (ccw) orientation.
Thus, by Stoke's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl}\vec{F} \cdot \vec{k}\, dS \leftarrow (+1)$$

$$= \iint_D z+1\, dA \leftarrow (+1)$$

$z=0$

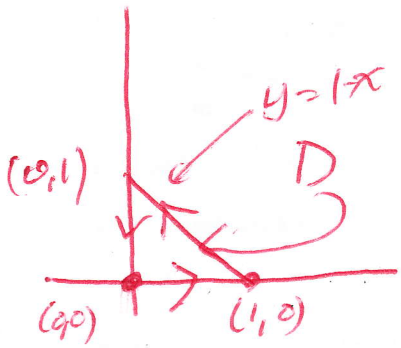
$$= \iint_D dA = \text{Area}(D) = \pi$$

(+1) (+1)

4. (8 points) Let C be the triangular path given by starting at $(0,0)$, then going to $(1,0)$, then to $(0,1)$ and finally returning to $(0,0)$. Let $\vec{F} = \langle e^{x^2} - y, \sin y + x^2 \rangle$. Use Green's theorem to compute

$$\oint_C \vec{F} \cdot d\vec{r}.$$

+1



Let $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

The C has positive orientation about D +3

Therefore, by Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$ +2

$$= \int_0^1 \int_0^{1-x} (2x+1)\, dy\, dx.$$

$$= \int_0^1 (2x+1)(1-x)$$

$$= \int_0^1 -2x^2 + x + 1$$

$$= \frac{-2}{3} + \frac{1}{2} + 1$$

$$= +5/6$$

+2