Math 120 KAUST    Final Exam (80 minutes)    September 27th, 2018

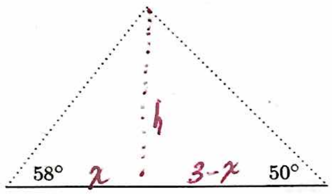
Name: Key

Directions:

- You have 80 minutes to complete this exam.
- There are 4 problems on 9 pages.
- Only TI 30 calculators are allowed.
- You are allowed one hand-written sheet (two sided is ok) of notes on regular 8.5-11 paper.
- You must show ALL your work.
- Leave answers in EXACT FORM or record up to 2 DECIMAL PLACES.
- If you need extra room, use the back side of the page. Include a note to indicate to the reader that you have done so.
- If you have any questions, raise your hand. In particular, if you are confused about the wording of a question raise your hand and I will try my best to answer.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 25 | |
| Total: | 70 | |

1. (10 points) Hunter and Oscar were camping 3 miles apart from each other in the Nevada desert, when they both spot a UFO flying high in the sky. Hunter measures it at an angle of 50° from the ground, and Oscar at 58°. How high above the ground is the UFO?




① $\tan(58^\circ) = \frac{h}{x} \Rightarrow h = x\tan(58^\circ)$ (+5)

② $\tan(50^\circ) = \frac{h}{3-x} \Rightarrow h = (3-x)\tan(50^\circ)$ (w/ x & 3-x

(+3) eq

$$x\tan(58^\circ) = (3-x)\tan(50^\circ)$$

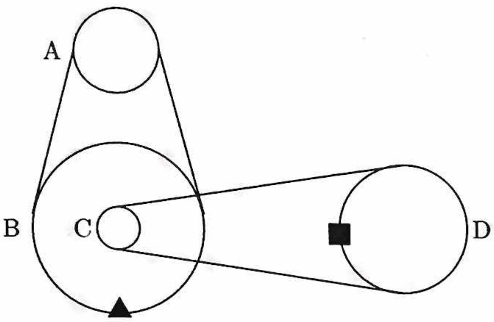
$$x\tan(58^\circ) + x\tan(50^\circ) = 3\tan(50^\circ)$$

$$x = \frac{3\tan(50^\circ)}{\tan(58)^\circ + \tan(50)^\circ}$$

$$h = x\tan(58^\circ) = \frac{3\tan(50^\circ)\tan(58^\circ)}{\tan(58^\circ) + \tan(50^\circ)}.$$

$$\approx 2.049 \text{ miles}$$

2. Below is a belt and wheel diagram, with each wheel labeled by a capital letter. The radii of wheels A,B,C, and D are 5,8,2, and 6 centimeters respectively. At the bottom edge of wheel B is a triangle sticker, and on the left edge of wheel D is a square sticker. These move with the rotations of the wheels, but not along the belts.




(a) (10 points) Set the origin as the center of wheel B, and suppose that the center of wheel D is 40 centimeters to the right. At $t = 0$ wheel A begins rotating counterclockwise 3 times a minute. Find parametric equations of motion for BOTH the triangle and the square $t$ seconds after wheel A begins rotating. (REMINDER, $t$ should be in *seconds*).

$\omega_A = 3\frac{rot}{min} = \frac{1}{20}\frac{rot}{sec} = \frac{\pi}{10}\frac{rad}{sec}$

| | $v$ | $r$ | $\omega$ |
|---|---|---|---|
| A | $\pi/2$ | 5 | $\pi/10$ |
| B | $\pi/2$ | 8 | $\pi/16$ |
| C | $\pi/8$ | 2 | $\pi/16$ |
| D | $\pi/8$ | 6 | $\pi/48$ |

+5

**Triangle** +2

$R = 8$

$\theta_0 = -\pi/2$ or $3\pi/2$

$\omega = \pi/16$

$(h,k) = (0,0)$

$\Downarrow$

$x_\Delta(t) = R\cos(\omega t + \theta_0) + h$

$= 8\cos(\frac{\pi}{16}t - \pi/2)$

$y_\Delta(t) = 8\sin(\pi/16 - \pi/2)$

**Square** +3

$R = 6$

$\theta_0 = \pi$ or $-\pi$

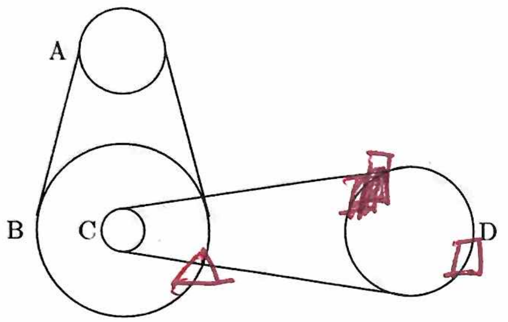
$\omega = \pi/48$

$(h,k) = (40,0)$

or ...

$\Rightarrow x_\square(t) = 6\cos(\frac{\pi}{48}t + \pi) + 40$

$y_\square(t) = 6\sin(\frac{\pi}{48}t + \pi)$




(b) (6 points) Above is the same belt and wheel apparatus. Notice that the triangle and square are missing. It is your job to fill them in. Mark above the locations of the triangle and square at $t = 39$ seconds.

Easiest to compute angle

$\theta_\Delta(t) = \frac{\pi}{16}t - \frac{\pi}{2}$

$\theta_\Delta(39) = \frac{39\pi}{16} - \frac{\pi}{2} = \frac{39\pi}{16} - \frac{8\pi}{16} = \frac{31\pi}{16} \Longrightarrow$

$\theta_\square(t) = \frac{\pi}{48}t + \pi$

$\theta_\square(39) = \frac{39\pi}{48} + \frac{48\pi}{48} = \frac{87\pi}{48}$

~~position~~ $\Longrightarrow$

(c) (4 points) What is the distance in centimeters between the square and the circle at $t = 39$ seconds.

$x_\Delta(39) = 8\cos\left(\frac{31\pi}{16}\right) = 7.846$

$y_\Delta(39) = 8\sin\left(\frac{31\pi}{16}\right) = -1.561$

$x_\square(39) = 6\cos\left(\frac{87\pi}{48}\right) + 40 = 44.989$

$y_\square(39) = 6\sin\left(\frac{87\pi}{48}\right) = -3.333$

$\text{Dist} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(44.989 - 7.846)^2 + (-3.333 + 1.561)^2}$

$= 37.185$

3. The high price of diamonds is derived from relative rarity of the stone. Indeed, if more and more diamonds pour into the market, the price of a 1-carat diamond will drop and drop, but of course never below 0. We therefore model the price of diamonds as a linear fractional transformation $f(x)$ where $x$ is the number of diamonds on the market.

(a) (6 points) When there were only about 5,000 diamonds on the market, the price was about \$100,000 each. Now there are closer to 200,000 diamonds on the market and the price is about \$4,000. Find an equation for $f(x)$.

$f(x) = \frac{ax+b}{cx+d}$. H.A. $y=0$. $\Longrightarrow a=0$

$c=1$ WLOG

$f(x) = \frac{b}{x+d}$

$(5000, 100000)$: $\frac{b}{5000+d} = 100000 \Longrightarrow b = 500000000 + 100000d$

$(200000, 4000)$ $\frac{b}{200000+d} = 4000 \Longrightarrow b = 800000000 + 4000d$

So $5 \times 10^8 + 100000d = 8 \times 10^8 + 4000d$

$96000d = 3 \times 10^8$

$d = 3125$

$b = 5 \times 10^8 + 100000(3125)$

$= 500000000 + 312500000$

$= 812500000$

So $f(x) = \frac{812500000}{x+3125}$

(b) (5 points) Write a function $g(x)$ which will output the number of diamonds on the market if their current price is $x$. What is the domain of this function? (Assume $f$ from part (a) has domain: $x > 0$).

If $f(x) = \frac{ax+b}{cx+d}$ then $f^{-1}(x) = \frac{-dx+b}{cx-a}$

so $g(x) = f^{-1}(x) = \frac{-3125x + 812500000}{x}$

Domain $0 < x \leq f(0)$

$\Downarrow$

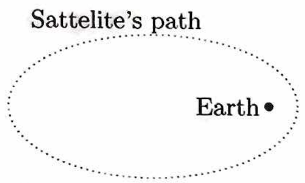
$0 < x \leq \frac{812500000}{3125} = 260000$

(c) (4 points) Leonardo can make artificial diamonds, and wants to flood the market with diamonds so that they become almost worthless. How many diamonds would Leonardo have to make in order for diamonds to be worth less than \$1000? What about less than \$100? (Don't forget that there are already 100,000 diamonds on the market).

$g(1000) = 809375$ | Subtract 100 000 $\Longrightarrow$ 709375

$g(100) = 8121875 \Longrightarrow 8021875$

4. A satellite has a slow elliptical orbit around planet Earth (see diagram below, don't worry, you will not need to model this path).




The furthest reaches of the orbit take the satellite 30,000 miles from Earth, while the closest point of the orbit to Earth is 1000 miles. The orbit is quite slow, taking 40 years to complete a full orbit.

(a) (7 points) The distance of the satellite from earth after $t$ years can be modeled as a sinusoidal function $d(t)$. Suppose the ~~rocket~~ is at its closest to earth (1000 miles) at $t = 0$. Write the rule for the function $d(t)$. satellite

$$A = \frac{max - min}{2} = \frac{30000 - 1000}{2} = 14500$$

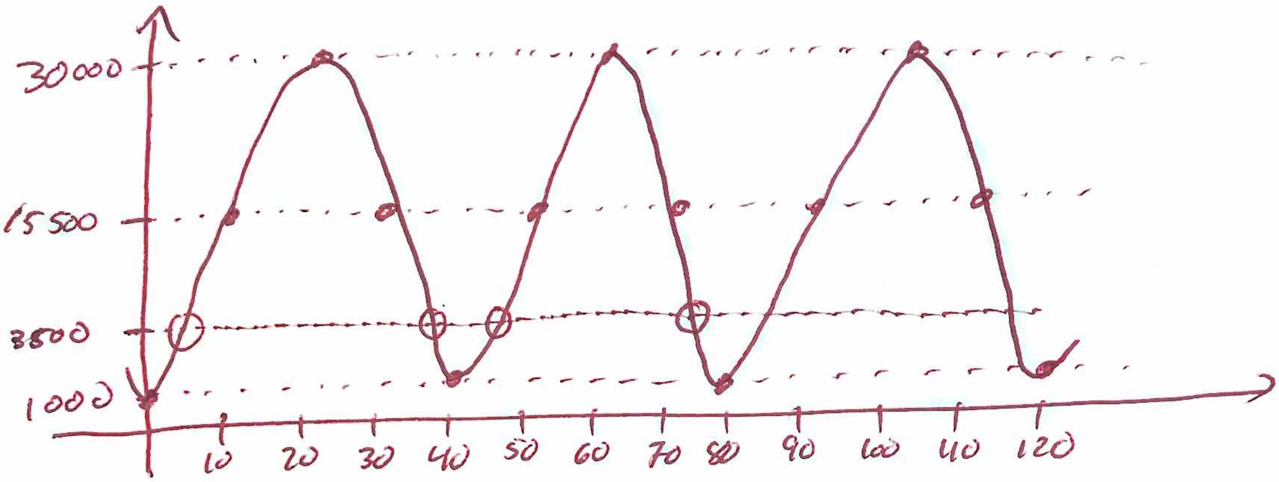
$$B = 40 \qquad B/4 = 10$$

$$C = (x - min) + B/4 = 10$$

$$D = \frac{max + min}{2} = \frac{31000}{2} = 15500$$

$$d(t) = 14500\sin\left(\frac{2\pi}{40}(t-10)\right) + 15500$$

(b) (5 points) Sketch a graph of $d(t)$ during the satellite's first 100 years in orbit.




(c) (8 points) The satellite must be within 3500 miles to be able to communicate with earth. List the first 4 times the satellite is exactly 3500 miles from earth.

$$14500\sin\left(\frac{2\pi}{40}(t-10)\right)+15500=3500$$

$$\sin\left(\frac{2\pi}{40}(t-10)\right)=-.8276$$

**Principal**

$$\frac{2\pi}{40}(t-10)=\text{Arcsin}(-.8276)$$

$$=-.9748$$

1st) $t \approx 3.7941$

$+B$

3rd) $t \approx 43.7941$

**Symmetry**

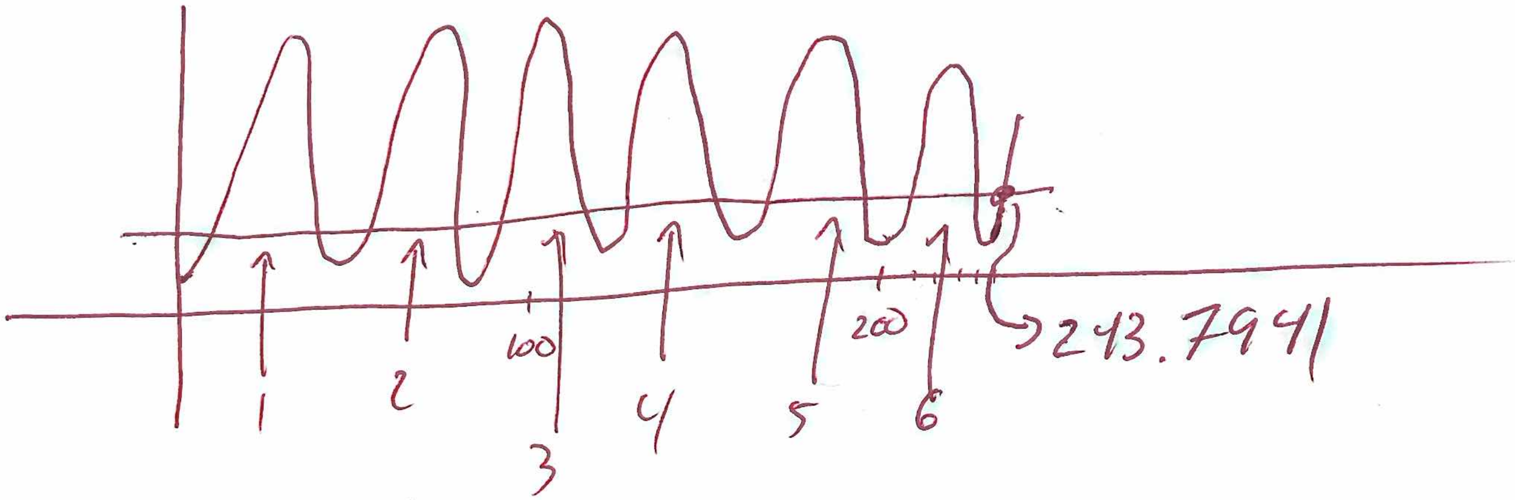
$$\frac{2\pi}{40}(t-10)=\pi-\arcsin(-.8276)$$

2nd) $t \approx 36.2059$

4th) $t \approx 76.2059$

(d) (5 points) In the first 250 years of orbit, how much time does the satellite spend out of contact with earth.

Sketch




Each orbit, out of contact for

36.2059 − 3.7941 = 32.4118 years

6 orbits ⟹ 194.4708 years

+ last bit 250 − 243.7941 = 6.2059 yrs

Total: 200.6767 years