Nov3

Monday, November 2, 2020 10:36 PM

Setup

Ω = set of outcomes.

$$Pr : \Omega \longrightarrow \mathbb{R}$$

$Pr(\omega)$ = Probability that ω occurs.

Events $E \subset \Omega$.

$$Pr(E) = \sum_{\omega \in E} Pr(\omega).$$

Random Variables

Example Ω = rolls of 2 dice (ordered).

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{Z} \\ (n,m) &\longmapsto n+m \end{aligned}$$

$\left(\begin{array}{l} X(\text{roll}) = \text{sum of value} \\ \quad\text{of 2 dice} \end{array} \right)$

$$E = \text{"roll a 7"} = \{\omega \in \Omega \mid X(\omega) = 7\}$$

Upshot:

X parametrizes all possible values of dice rolls into one function.

Get all these events as preimages of points under this map X.

Def$^{\underline{n}}$ A random variable is a function

$$X : \Omega \longrightarrow \mathbb{R}.$$

(or $X : \Omega \longrightarrow S$ ← set of ciphers or some other set)

upshot Binds together outcomes into events in a coherent way (according to fn X).
"Get families of events"

Given a random variable $X:\Omega\longrightarrow\mathbb{R}$ get 3 events for every $\underline{x}\in\mathbb{R}$.

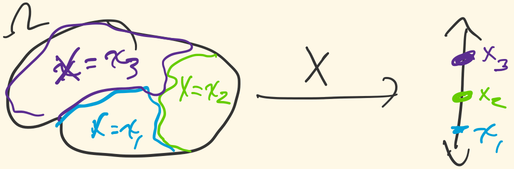
$$\longrightarrow \{\omega \mid \underline{X(\omega)=x}\} =: X=x$$

$$\{\omega \mid X(\omega)\le x\} \leftarrow$$

$$\{\omega \mid X(\omega)\ge x\}.$$

Defn $f:\Omega\longrightarrow\mathbb{R}$ be a random variable. The probability density function of X is (for $x\in\mathbb{R}$)

$$f_X(x) = Pr(X=x)$$
$$= Pr(\omega \mid X(\omega)=x).$$



Lemma If $x\neq y \implies X=x$ is disjoint from $X=y$.

Pf/ Exercise.

Examples

* Uniform Distribution

$$X : \Omega \longrightarrow S \subset \mathbb{R}$$

$$S = \{x_1, \ldots, x_n\}$$

$$f_X(x_i) = \frac{1}{n}$$

$$= \Pr(X = x_i) = \Pr(\{\omega \mid X(\omega) = x_i\})$$

Then X is a uniform distribution of Ω.

* Binomial Distribution

Setup An experiment w/ 2 outcomes.
Success or failure.

→ p = probability of success.

Ω = {outcomes of running n times}.

$$X : \Omega \longrightarrow \mathbb{R}$$

$$\omega \longmapsto \#\text{ of successes in } \omega.$$

$$\Omega = \{\omega = b_1 b_2 \cdots b_n \mid b_i \in \underset{\text{failures}}{0}, \underset{\text{success}}{1}\}$$

$$X(b_1 b_2 \cdots b_n) = b_1 + b_2 + \cdots + b_n.$$

Take $\omega \in \Omega$. # succes # failures.

$$\Pr(\omega) = \underset{\substack{\uparrow \\ \text{prob} \\ \text{of success}}}{p}^{X(\omega)}(1-\underset{\substack{\uparrow \\ \text{prob of} \\ \text{failure}}}{p})^{n-X(\omega)} \quad \leftarrow$$

Question: what is prob I get exactly k successful outcomes?

$$f_X(k) = \Pr(X = k)$$

$$= \sum_{\substack{\omega \in \Omega \\ X(\omega)=k}} \Pr(\omega)$$

$$= \sum_{\substack{\omega \text{ s.t.} \\ \underline{X(\omega)=k}}} p^{X(\omega)}(1-p)^{n-X(\omega)}$$

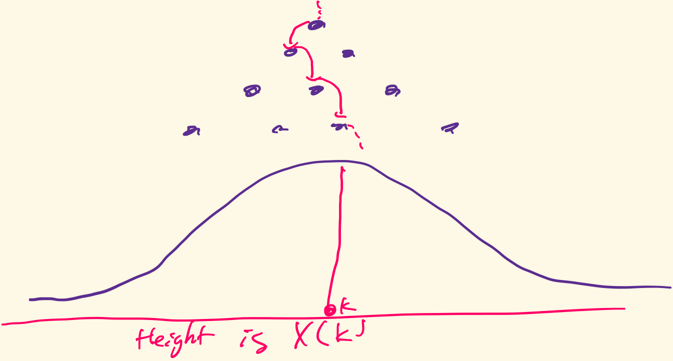
$$= \sum_{\substack{\omega \text{ s.t.} \\ X(\omega)=k}} \underline{p^k(1-p)^{n-k}}$$

$$= p^k(1-p)^{n-k} \sum_{\substack{\omega \text{ s.t.} \\ X(\omega)=k}} 1$$

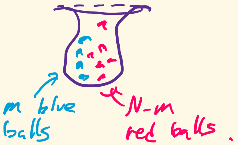
$$= \underline{\#\{\omega \mid X(\omega)=k\}} \cdot p^k(1-p)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Ex $p = \frac{1}{2}$ $\rightsquigarrow$ Bell curve $\rightsquigarrow$ $\binom{n}{k} \cdot \frac{1}{2^n}$



Hypergeometric Distribution



ω = way to pick n balls

X(ω) = # blue balls chosen.

$$i, \quad f_X(i) = \Pr(\text{choose } i \text{ blue balls}).$$

$$= \frac{\binom{m}{i} \cdot \binom{N-m}{N-i}}{\binom{N}{n}}$$

Geometric Distribution

(Example of ∞ Ω).

A (potentially unfair) coin which hits Heads w/ prob $0 < p < 1$

$\omega \in \Omega$ is an infinite sequence of tosses.

$$\cdots \quad \Omega = \{b_1 b_2 b_3 \cdots \mid b_i = \underset{\text{tails}}{0}, \underset{\text{Heads}}{1}\}$$

$E = \{\omega \in \Omega \mid \omega \text{ starts w/ } \underline{b_1 b_2 \cdots b_n}\}.$

$Pr(E) = p^{\# b_i = 1} \cdot (1-p)^{\# b_i = 0}$

observation is later coin flips don't matter so we can pretend only flip n-times.

$X : \Omega \longrightarrow R$

$X(\omega) = \min \{i \text{ s.t. } b_i = 1\}.$

= first time I flip heads.

$(X=k) = \{\omega \mid X(\omega) = k\} = \{\omega = \underbrace{0000\cdots0}_{k-1} 1\, b_{k+2} \cdots \}$

$f_X(k) = Pr(X=k) = p(1-p)^{k-1}$

Rmk Called geometric b/c $\dfrac{f_X(k+1)}{f_X(k)} = 1-p \quad \forall k$

So sequence $f_X(k), f_X(k+1), f_X(k+2), \ldots$

Before we were talking about comparing various events.

One variable gives disjoint events, so to generalize that conversation we need multiple variables.

Def^n

$X, Y : \Omega \longrightarrow R$ 2 random variables.

The joint density function is

$f_{X,Y} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$$f_{X,Y}(x,y) = Pr(X=x \text{ and } Y=y).$$

$$= Pr(\{\omega \mid X(\omega)=x \text{ and } Y(\omega)=y\})$$

The conditional density function is

$$f_{X|Y}(x|y) = Pr(X=x \mid Y=y)$$

$$= Pr(\{\omega \mid X(\omega)=x\} \mid \{\omega \mid Y(\omega)=y\})$$

X and Y are Independent if

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \longleftarrow$$

Equiv $(X=x)$ & $(Y=y)$ are indep events for every $x,y \in \mathbb{R}$.

Lemma $f_{X,Y}(x,y) = f_X(x)\cdot f_{Y|X}(y|x) \longleftarrow$

Pf/ $E=(X=x)$ $\quad F=(Y=y)$

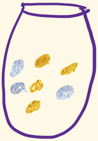
$$Pr(E\cap F) = Pr(E)\, Pr(F|E)$$

↑ Last time. $\emptyset$

Cor X,Y are independent

$$\Longleftrightarrow f_{X|Y}(x|y) = f_X(x) \quad \forall x,y \in \mathbb{R}.$$

Example



4 gold coins
3 silver coins.

*Take a coin.
Examine
→ (and return)* ←
*Take a second
coin & examine.

Define 2 random variables

X = # gold coins.

Y = # silver coins.

$$f_{X,Y}(x,y) = \Pr(x \text{ gold coins } \& \ y \text{ silver coins}).$$

Break it into the 2 separate picks.
Do formally by introducing 2 random variables

$$F = \begin{pmatrix} \text{\# golds on} \\ \text{first pick} \end{pmatrix} = \begin{cases} 1 & \text{first gold} \\ 0 & \text{first silver.} \end{cases}$$

$$S = \begin{pmatrix} \text{\# golds on} \\ \text{second pick} \end{pmatrix} = \begin{cases} 1 & \text{second gold} \\ 0 & \text{second silver.} \end{cases}$$

Observe

① $X = F + S$

② $Y = 2 - X = 2 - F - S$

③ $\left.\begin{aligned} \Pr(F=1) &= 4/7 \\ \Pr(S=1) &= 4/7 \end{aligned}\right\}$ easy to compute.

Want

$$f_{X,Y}(1,1) = \Pr(X=1 \text{ and } Y=1) = \Pr\left[\begin{matrix} F=0 \text{ and } S=1 \\ \text{or} \\ F=1 \text{ and } S=0 \end{matrix}\right]$$

disjoint

$$= \Pr(F=0 \text{ and } S=1) + \Pr(F=1 \text{ and } S=0)$$

$$\circledast \quad = \Pr(F=0)\Pr(S=1) + \Pr(F=1)\Pr(S=0)$$

$$= \frac{3}{7}\cdot\frac{4}{7} + \frac{4}{7}\cdot\frac{3}{7} = .4898.$$

If we don't return coin

$$f_X(1,1) = \cdots = \Pr(F=0 \;\&\; S=1) + \Pr(F=1 \;\&\; S=0)$$

$$= \Pr(F=0)\cdot\Pr(S=1 \mid F=0) + \Pr(F=1)\cdot\Pr(S=0 \mid F=1).$$

$$= \frac{3}{7}\cdot\frac{4}{6} + \frac{4}{7}\cdot\frac{3}{6} = \frac{4}{7} \approx .5714$$

Thm Bayes' for random variables

X, Y random variables $f_Y(y) > 0$.

$$f_{X|Y}(x/y) = \frac{f_X(x)\cdot f_{Y|X}(y|x)}{f_Y(y)}.$$

Examples

Explore independence of (X, Y, Z)

s.t. $X, Y : \Omega \longrightarrow \pm 1$

1 half time -1 other half.

X, Y are indep.

Define $Z = XY$. (also $\Omega \to \pm 1$)

Question $$f_Z(1) = \sum_{\substack{x \in \{\pm 1\} \\ y \in \{\pm 1\}}} \underbrace{\Pr(Z=1 \mid X=x \text{ and } Y=y)} f_{X,Y}(x,y).$$

$$= \begin{cases} 0 & x \neq y \leftarrow (-1)\cdot 1 = -1 \text{ so } Z \neq 1 \text{ ever} \\ 1 & x = y \leftarrow (-1)(-1) \text{ or } 1\cdot 1 \Rightarrow Z = 1 \end{cases}$$

$$= f_{X,Y}(1,1) + f_{X,Y}(-1,-1)$$

$$= f_X(1) f_Y(1) + f_X(-1) f_Y(-1)$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{2}.}$$

When are Z & X both = 1

$$f_{ZX}(1,1) = \Pr((Z=1) \text{ and } (X=1))$$

$$= \Pr(Y=1 \text{ and } X=1)$$

$$= f_X(1)\cdot f_Y(1)$$

$$= 1/4$$

$$= f_Z(1)\cdot f_X(1) \checkmark$$

Check same way for all other pair

get $f_{Z,X}(z,x) = f_Z(z)\cdot f_X(x)$

$\Longrightarrow Z, X$ are indep.

Also show Z, Y are indep.

But X, Y, Z are not. (independence is not pairwise).

Defn

A family $X_1, \dots, X_n$ of random variables is independent if $\forall\, x_1, \dots, x_n$ the events $\{X_1 = x_1$ and $X_2 = x_2$ and ... and $X_n = x_n\}$ are independent.

Example

$$Pr(Z=1 \;\&\; X=1 \;\&\; Y=-1) = 0$$

$$\neq$$

but

$$Pr(Z=1)\cdot Pr(X=1)\cdot Pr(Y=1) = \frac{1}{8}$$

Expected Value

Ex

$X : \Omega^{\text{dice}} \longrightarrow \mathbb{R}$ value of roll.

What is X on average?

by prob.

Expected value is a weighted average of all values a variable can take

Def^n $X : \Omega \to \mathbb{R}$ random variable.

X takes values $x_1, \dots, x_n$.

The expected value of X is

$$E(X) = \sum_{i=1}^{n} x_i f_X(x_i) = \sum x_i \Pr(X = x_i)$$

Example $X : \Omega \to \mathbb{Z}$ sum of dice.

$$E(X) = \sum_{i=2}^{12} i \Pr(X = i)$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \dots$$

$$= 7.$$

Rmk The word expected is misleading.

⚠ **Ex** 1 die what do I expect to roll.

$$E(X) = \sum_{i=1}^{6} i (\Pr(X = i))$$

$$= \sum_{i=1}^{6} i \cdot \frac{1}{6} = 3.5$$