March30

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Recall:

Theorem (Recognizing direct products)

Let $H, K \le G$ Suppose

① $H, K \trianglelefteq G$

② $H \cap K = 1$

Then $HK = \{hk : h \in H, k \in K\} \simeq H \times K$.

Key ideas:

Lemma A

If $H \cap K = 1$ then every $x \in HK$ can be written uniquely as $x = hk$ for $h \in H$, $k \in K$.

Consequence Get a bijection

$$HK \xleftrightarrow{\sim} \{(h, k) : h \in H, k \in K\}$$

$$hk \longleftarrow\!\shortmid (h, k)$$

Use condition ① to show this is a hom.

Upshots (in situation of thm)

① Algebra of HK (a priori depends on G) only depends on H & K.

② $HK/H \simeq K$ $\qquad HK/K \simeq H$.

→ ③ $h \in H,\ k \in K$ $\qquad hk = kh$.

④ If $HK = G$, decomposed G into smaller groups.

Question What happens (in coords) if weaken condition ①.

Setup $H, K \le G$.

① $H \trianglelefteq G$ (so HK still a subgroup).

② $H \cap K = 1$

Lemma A tells us (condition ②)

$$HK \xleftrightarrow{\text{bijection.}} \{(h,k) : h \in H,\ k \in K\}$$

↑ as a set!! Multiplication not coordinatewise.

$$x_1 = h_1k_1 \longleftarrow\!\!| \ (h_1, k_1)$$

$$x_2 = h_2k_2 \longleftarrow\!\!| \ (h_2, k_2)$$

Multiply

$$x = x_1x_2 \longmapsto (h,k) \leftarrow \text{what is this?}$$

$\quad \|$

hk ←

unique $h\in H$, $k\in K$

Posed Carefully

What does multiplication in HK correspond to on the right?

Calculations ($\dagger$)

$$x_1x_2 = h_1k_1 \cdot h_2k_2$$

$$= h_1k_1h_2(k_1^{-1}k_1)k_2$$

$$= h_1\underbrace{(k_1h_2k_1^{-1})}_{\in H\trianglelefteq G}k_1k_2$$

$$= (h_1\, k_1{*}h_2)(k_1\, k_2)$$

Action of $K \circlearrowleft H$ by conjugation (in G).

Translating to right side.

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1{*}h_2), k_1k_2) \quad (\dagger\dagger)$$

Multiplication in HK depends on H, K and an action $K \circlearrowleft H$ By actomorphism

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & S_H \arrow[d, phantom, sloped, "\supseteq"] \\ & \mathrm{Act}\, H \end{tikzcd}$$

Observation The action $K \circlearrowleft H$ is by automorphisms.

Pf/ $k \in K$ $\quad \sigma_k : H \longrightarrow H$ is a homomorphisms

$$\sigma_k(h_1h_2) = kh_1h_2k^{-1}$$
$$= kh_1k^{-1}kh_2k^{-1}$$
$$= \sigma_k(h_1)\cdot\sigma_k(h_2)$$

Def^n (Semidirect Products).

Let H, K be groups. Let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be a homomorphism defining an action $K \circlearrowright H$ by automorphisms: $k * h = \phi(k)(h)$

Let $G = \{(h,k) : h \in H, k \in K\}$ Define multiplication in G as in $(\dagger\dagger)$

$$(h_1,k_1)(h_2,k_2) = (h_1(k_1 * h_2), k_1k_2) \quad (\dagger\dagger)$$

Then G is called the semidirect product of H & K with respect to ϕ & denoted

$$G = H \rtimes_\phi K$$

When ϕ is understood $G = H \rtimes K$.

Theorem Fix $\phi : K \longrightarrow \mathrm{Aut}(H)$
Set $G = H \rtimes K$. Then

① G is a group w/ $|G| = |H| \cdot |K|$.

② $H \xrightarrow[\sim]{} \{(h,1) : h \in H\} \leq G$ ←

$K \xrightarrow[\sim]{} \{(1,k) : k \in K\} \leq G$

Rmk Identify H, K as subgroups of G.

③ $H \trianglelefteq G$. $G/H \simeq K$.

④ $H \cap K = 1$

⑤ $k \in K \leq G$. $khk^{-1} = k * h$.
$h \in H \leq G$.

Proof ① HW.

(a) Identity is $(1,1)$

(b) $(h,k)^{-1} = (k^{-1} * h^{-1}, k^{-1})$ ←

(c) use action by acts.

② Let $\tilde{H} = \{(h,1)\}$

$$(a,1)(b,1) = (a\ (1*b), 1)$$
$$= (ab, 1).$$

$$(a,1)^{-1} = (1^{-1} * a^{-1}, 1) = (a^{-1}, 1).$$

so $\tilde{H} \leq G$.

Then $H \longrightarrow G$ injective hom w/ image $\tilde{H}$

✓

Let $\tilde{K} = \{(1,k)\}$

$\sigma_c : H \longrightarrow H$ is an act

$\sigma_c(1)=1$

$$(1,c)\cdot(1,d) = (1\cdot \underbrace{(c*1)}, cd)$$

$$= (1, cd).$$

Rest is same.

$$(1,c)^{-1} = (c^{-1}*1, c^{-1}) = (1, c^{-1}).$$

$$\Rightarrow \tilde{K} \le G.$$

④ Obvious $\quad H\cap K = \{(h,1)\}\cap\{(1,k)\} = (1,1).$

③
$$\begin{tikzcd} G \arrow[r, "\pi"] & K \\ (h,k) \arrow[r, mapsto] & k \end{tikzcd}$$

surjective ✓

Homom

$$\pi((h_1,k_1)(h_2,k_2)) = \pi(h_1 k_1*h_2, k_1k_2) = k_1k_2$$

$$\pi(h_1,k_1)\pi(h_2,k_2) = k_1k_2 \quad ✓$$

kernel is H.

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⑤ $k\in K \qquad h\in H$

$$khk^{-1} = \underbrace{(1,k)(h,1)}(1,k)^{-1}$$

$$= (k*h, k)(1,k^{-1})$$

$$= ((k*h)\underbrace{(k*1)}_{1}, \underbrace{kk^{-1}}_{1})$$

$$= (k * h, 1) = k * h.$$ ∅

Remarks

* $H \rtimes K$ emphasizes $H \trianglelefteq G$.

* This is not symmetric in general.

$$\underset{\substack{\uparrow \\ K \to \operatorname{Aut} H}}{H \rtimes K} \not\cong \underset{\substack{\uparrow \\ H \to \operatorname{Aut} K.}}{K \rtimes H}$$

Theorem (Recognizing Semidirect Products)

$H, K \le G$. Assume

① $H \trianglelefteq G$

② $H \cap K = 1$

Then $HK \cong H \rtimes_\phi K$ where

$\phi : K \longrightarrow \operatorname{Aut}(H)$ is the perm rep of the conjugation action of K on H in G.

Proof

$$\begin{aligned} H \rtimes_\phi K &\longrightarrow HK \\ (h, k) &\longmapsto hk \end{aligned}$$ bij. by Lemma A

Homomorphism by (+). ∅

Direct	Semidirect	
H, K	$H, K, \phi: K \longrightarrow Aut(H)$	
$H \times K$	$H \rtimes_\phi K$	external
$H, K \trianglelefteq G$ $H \cap K = 1$	$H \trianglelefteq G$ $H \cap K = 1$	Internal

Remark Direct Product is a special case of semidirect product:

Let
$$\begin{aligned} \phi: K &\longrightarrow Aut(H) \\ k &\longmapsto id_H \end{aligned}$$

Then in $H \rtimes_\phi K$. what is mult?

$$(h_1, k_1)(h_2, k_2) = (h_1 \underbrace{(k_1 * h_2)}_{h_2}, k_1 k_2) = (h_1 h_2, k_1 k_2).$$

There is always a bijective map

$$\begin{aligned} \Phi: H \rtimes_\phi K &\longrightarrow H \times K \\ (h, k) &\longmapsto (h, k) \end{aligned}$$

Not always a homomorphism.

Theorem

Let H, K be groups & $\phi : K \longrightarrow Aut(H)$.

The following are equivalent.

① Φ is a homomorphism (isomorphism).

② ϕ is the trivial map.

③ $K \trianglelefteq H \rtimes K$.

Proof

①$\Longrightarrow$② $\quad \begin{matrix} x = (h_1, k_1) \\ y = (h_2, k_2) \end{matrix} \in H \rtimes K$.

$$xy = (h_1 \; k_1 * h_2, \; k_1 k_2)$$

$$\Phi(xy) = \Phi(x)\Phi(y) = (h_1 h_2, \; k_1 k_2)$$

Φ is identity on set $\Longrightarrow$

$$h_1 \; k_1 * h_2 = h_1 h_2$$

$$\longrightarrow \quad k_1 * h_2 = h_2 \qquad \forall \; \begin{matrix} k_1 \in K \\ h_2 \in H \end{matrix}$$

Vary k_1 & h_2, see action is trivial so ϕ is trivial map.

②$\Longrightarrow$① Remark above.

$\begin{matrix} x = (h_1, k_1) \\ y = (h_2, k_2) \end{matrix} \in H \rtimes K \qquad xy = (h_1 h_2, \; k_1 k_2)$

. So $\Phi(xy) = \Phi(x)\Phi(y)$

②⇒③ Fix $k \in K$, $h \in H$

$$\begin{aligned} kh &= (1,k)(h,1) \\ &= (k*h, k) \\ &= (h,k) \\ &= (h,1)(1,k) = hk \end{aligned}$$

$hkh^{-1} = k$

So $H \le N_G(K)$. Also $K \le N_G(K)$

So $\underset{G}{\underset{\shortparallel}{HK}} \le N_G(K) \Rightarrow K \trianglelefteq G.$

③⇒②

If $K \trianglelefteq G$. Know $H \trianglelefteq G$ & $H \cap K = 1$

(Thm)

by recognition for direct products, if $h \in H$, $k \in H$ then $hk = kh$.

So $\underset{k*h}{\underset{\shortparallel}{khk^{-1}}} = h$ So action trivial.

∅

Corollary

If $\phi : K \longrightarrow \mathrm{Aut}\, H$ is nontrivial, then $H \rtimes_\phi K$ is not abelian.

Pf/

Pf 1 If ϕ nontrivial $\Longrightarrow$ K not normal $\Longrightarrow$ $H \rtimes_\phi K$ not abelian.

Pf 2

$\phi \neq 1$ $\exists$ $h \in H$, $k \in K$ s.t. $h * k \neq h$.

$$khk^{-1} \neq h$$

$$kh \neq hk.$$

Rmks

Vast new fountain of interesting & exotic groups. So far:

1) Cyclic group & their products.
2) Groups from geometry.
   * D_{2n}
   * $GL_n(F)$, $SL_n(F)$, T, ...
3) Permutation groups.
   * S_n, A_n, ...
4) Multiplication Tables for small groups
   * Q_8

Examples

① Z_2 extensions of abelian groups.

Fact A an abelian group.

$$\begin{tikzcd} i: A \arrow[r] & A \\ a \arrow[r, mapsto] & a^{-1} \end{tikzcd}$$

Check this is a homomorphism.

Define

$$\begin{tikzcd} \phi: Z_2 \arrow[r] & \mathrm{Aut}(A) \\ 1 \arrow[r, mapsto] & id_A \\ x \arrow[r, mapsto] & i \end{tikzcd}$$

Form $A \rtimes_\phi Z_2$ nonabelian group.

② Do this for $A \simeq Z_n$.

Claim $Z_n \rtimes Z_2 \simeq D_{2n}$.

Proof

$Z_n = \langle y \rangle$, $Z_2 = \langle x \rangle$

$$\begin{tikzcd} D_{2n} \arrow[r] & Z_n \rtimes Z_2 \\ r \arrow[r, mapsto] & y \\ s \arrow[r, mapsto] & x \end{tikzcd}$$

$$\left(\begin{array}{l} sr^i \mapsto yx^i \\ r^i \mapsto x^i \end{array} \right)$$

Must show relations of D_{2n} hold in target.

$r^n = 1 \longleftrightarrow y^n = 1$ ☺

$$s^2 = 1 \longleftrightarrow x^2 = 1$$ ☺.

$$rs = sr^{-1} \longleftarrow \underline{\underline{y}}\;x\underline{\underline{y}}^{-1} = (1,x)(y^{-1},1)$$

$$= (x * y^{-1}, x)$$

$$= (y, x)$$

$$= \underline{\underline{yx}}$$

✓

Bijective? Surjectivity is clear & groups are same size.

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③ Generalized inversion extensions.

abelian.

$$\langle x\rangle = \mathbb{Z}_{2n} \longrightarrow \mathrm{Aut}(A)$$

$$x \longmapsto \iota$$

$$x^i * a = \begin{cases} a & i \text{ even} \\ a^{-1} & i \text{ odd.} \end{cases}$$

Form $G = A \rtimes \mathbb{Z}_{2n}$.

Observe $x^2 \in Z(G)$.

$$xax^{-1} = a^{-1}$$

$$x^2 a x^{-2} = (a^{-1})^{-1} = a$$

$$x^2 \in C_G(A)$$

$x^2 \in C_G(Z_{2n})$

$x^2 \in C_G(\langle A, Z_{2n}\rangle)$

$= C_G(G) = Z(G)$.

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④ Do this for $A = Z_3$. $Z_{2n} = Z_4$.

Get $Z_3 \rtimes Z_4$ a nonabelian group of order 12.

Prop $A_4, D_{12}, Z_3 \rtimes Z_4$ are nonisomorphic.

Pf/ Look at Sylow 2-subgroups.

A_4: $P_1 = \langle (12)(34)\rangle \trianglelefteq A_4$ $\quad n_2(A_4) = 1$

$$\cong \; Z_2 \times Z_2.$$

$\not\cong$

D_{12}: $P_2 = \{1, r^3, s, sr^3\} \ntrianglelefteq D_{12}$ $\quad n_2(D_{12}) = 3$

$$\cong \; \underline{Z_2 \times Z_2}$$

$\not\cong$

$r^2 s r^{-2} = s r^{-4} = s r^2 \notin P_2$

$\left(n_2 \equiv 1 \bmod 2, \; n_2 \mid 3, \; 12 = 2^2 \cdot 3 \right)$

$Z_3 \rtimes Z_4$ $\quad P_3 \simeq \underline{Z_4} \ntrianglelefteq Z_3 \rtimes Z_4$ $\quad n_2(Z_3 \rtimes Z_4) = 3$

⑤ $G \curvearrowright G$ by conjugation

$$G \longrightarrow \mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$$

For inner semidirect product

$G \rtimes G$.

Next Time (Semidirect Product as a tool for classification).

Outline

① Sylow to produce $P \trianglelefteq G$, $Q \le G$

② Counting to show $PQ = G$

③ Lagrange $P \cap Q = 1$

Recognition $\Longrightarrow$ $G \simeq P \rtimes_{\phi} Q$

Reduced classifying G to classifying maps $\phi : Q \longrightarrow \mathrm{Aut}(P)$

↑ easier in practice.