March11

Thursday, March 11, 2021 2:01 PM

Recall:

Fix a finite group G. $|G| = p^{\alpha} m$ w/ $p \nmid m$

Interested in $P \leq G$ w/ $|P| = p^{\alpha}$.

↑ Called Sylow p-subgroups.

Defined $\mathrm{Syl}_p(G) = \{\text{set of such } P\}$

$n_p(G) = n_p = \# \mathrm{Syl}_p(G)$.

Sylow Theorems

① $\exists\ P \in \mathrm{Syl}_p(G)$

② If $P, Q \in \mathrm{Syl}_p(G)$ then $P = gQg^{-1}$ $\Big\}$ $\Longrightarrow$ $n_p = 1 \iff P \trianglelefteq G$

③ $n_p \equiv 1 \mod p$ and $n_p \mid m$

$(n_p = [G : N_G(P)])$

Groups of order 30

Prop: Let $|G| = 30. = 2 \cdot 3 \cdot 5$ (never simple)

Then $\exists\ H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$

Proof

Observe If $\exists\ H \leq G$ w/ $|H| = 15$

1) $[G:H] = 2 \implies H \trianglelefteq G$

2) $15 = 3 \cdot 5$ & $3 \nmid (5-1) = 4$

last time. $\implies H \simeq \mathbb{Z}_{15}$

Suffice to produce $H \leq G$ of order 15

Fix $P \in Syl_3(G)$ $Q \in Syl_5(G)$

order 3 order 5

$\rightsquigarrow P \cap Q = \{1\}$ (LG)

$$|PQ| = \frac{|P| \cdot |Q|}{|P \cap Q|} = \frac{3 \cdot 5}{1} = 15$$

If PQ a subgroup... we win!

$\rightsquigarrow$ If P or Q normal $\implies$ PQ a group.

Reduced to proving either P or Q normal

$\boxed{\text{Assume not}}$

$n_3 = \{\cancel{1}, \cancel{4}, \cancel{7}, 10, \cancel{13}, \dots\}$ $n_3 \mid 10$

$n_5 = \{\cancel{1}, 6, \cancel{11}, \cancel{16}, \dots\}$ $n_5 \mid 6$

*10 sylow 3 subgroups

*6 sylow 5 subgroups.

* 10 subs order 3 $\{1, \underbrace{x, x^2}_{\text{2 elts order 3}}\}$

$\rightsquigarrow$ 20 elts order 3 in G

* 6 subs order 5 $\{1, \underbrace{x, x^2, x^3, x^4}_{\text{4 elts order 5}}\}$

$\rightsquigarrow$ 24 elts order 5 in G

$\dashrightarrow$ 44 elts in G ← order 30

↯

either P or Q is normal!

Exercise

Can show P, Q both normal.

Let $|G| = 12 = 2^2 \cdot 3$

Then G has a normal subgroup of order 3, or else $G \cong A_4$

Never Simple

$(A_4 \trianglerighteq \{(12)(34), (13)(42)\})$

proof

$n_3 = \{1, 4, 7, \ldots\}$ divides 4

n_3 1 or 4

If $n_3 = 1$ done.

Else $\mathrm{Syl}_3(G) = \{P_1, P_2, P_3, P_4\}$

$G \curvearrowright \mathrm{Syl}_3(G)$ by conjugation.

Exercise This is faithful.

$$G \hookrightarrow S_4$$

$$P_i = \{1, \underbrace{x, x^{-1}}_{\uparrow\ 8 \text{ elts order } 3 \text{ in } G.}\}$$

Elements of order 3 in S_4

3 cycles: $(a\ b\ c) \in A_4$

$$|G \cap A_4| \geq 8$$

divides 12

$$\Longrightarrow G \cap A_4 = G = A_4 \quad ⊘$$

$12 = 2^2 \cdot 3$
$45 = 3^2 \cdot 5$
$63 = 3^2 \cdot 7$

useful to have large normal subgroups

Theorem

G a finite group of order $p^2 \cdot q$ where $p \neq q$. Then G either has a normal sub order p^2 or a normal sub order q.

Outline $P \in Syl_p(G)$ $Q \in Syl_q(G)$

$p > q$	$q > p$
$P \trianglelefteq G$	$Q \trianglelefteq G$ or $G \cong A_4$

Pf/ Exercise.

Recall

Normality is not always transitive.

$$K \trianglelefteq H \quad \& \quad H \trianglelefteq G$$

$$\not\Longrightarrow \quad K \trianglelefteq G.$$

Lemma *

Let G a finite group.
Let $H \trianglelefteq G$.
Let $P \in Syl_p(G)$.
If $P \trianglelefteq H \Longrightarrow P \trianglelefteq G$.

Pf/ Exercise

Groups of order 60

Theorem

Let $|G| = \overset{2^2 \cdot 3 \cdot 5}{60}$ & $\underline{n_5 > 1}$
Then G is simple.

Corollary A_5 is simple.

Pf/ $\langle (1\ 2\ 3\ 4\ 5) \rangle \neq \langle (1\ 3\ 2\ 4\ 5) \rangle$

2 subs of order 5 ϕ

Prove the theorem by contradiction.

Suppose $\exists\ H \trianglelefteq G$ nontrivial.

$$|H| = \{\cancel{1}, \cancel{2}, \underset{\uparrow}{\cancel{3}}, \underset{\uparrow}{\cancel{4}}, \underline{\cancel{5}}, \cancel{6}, \cancel{10}, \cancel{12}, \cancel{15}, \cancel{20}, \cancel{30}, \cancel{60}\}$$

Lemma 1

$5 \nmid |H|$

Proof

if $5 \mid |H|$, then $P \in Syl_5(H)$ order 5

If $\underline{P \trianglelefteq H} \implies P \trianglelefteq G$ by lemma. $\ast$

But $\underline{n_p(G) > 1}$ $\lightning$

Know $n_5(H) = \{1, 6, \cancel{11}, \dots\}$

If $n_5(H) = 6 \rightsquigarrow 6$ subs order 5 in H

$\rightsquigarrow 24$ elts order 5 in H

$\rightsquigarrow |H| \geq 25$

so $|H| = 30$

But $Q \trianglelefteq \mathbb{Z}_{15} \trianglelefteq H$

Abelian groups have unique Sylow p-subs.

$|Q| = 5$

Lemma $\ast$ $\implies Q \trianglelefteq G$ so $Syl_5(G) = \{Q\}$

$\cancel{5}$ $\lightning$

Next eliminate 2, 3, 4

Let $\bar{G} = G/H$.

If $|H| = 2, 3, 4 \rightsquigarrow |\bar{G}| = \{30, 20, 15\}$.

Lemma 2

$\exists \ \bar{P} \trianglelefteq \bar{G}$ w/ $|\bar{P}| = 5$

Pf/

$|\bar{G}| = 30 \rightsquigarrow \bar{P} \trianglelefteq \mathbb{Z}_{15} \trianglelefteq \bar{G}$

$\uparrow \ |\bar{P}| = 5$

Normal in G by lemma *

$|\bar{G}| = 20 = 2^2 \cdot 5 \Longrightarrow \bar{P} \trianglelefteq \bar{G}$ w/ $|\bar{P}| = 5$

$q > p$

$20 \not\equiv 2$

$|\bar{G}| = 15 \Longrightarrow \bar{P} \trianglelefteq \mathbb{Z}_{15} = |\bar{G}|$

$\bar{P} \cong \mathbb{Z}_5$

Lemma 3

$H \neq 2, 3, 4$

Pf/

$\bar{P} \trianglelefteq \bar{G} \xleftrightarrow{\text{4th is}} H \trianglelefteq P \trianglelefteq G$

$\pi : G \longrightarrow \bar{G}$

persists to $\pi : P \longrightarrow \bar{P}$ surjective

& kernel is H

so $\bar{P} \simeq P/H$.

$$|P| = |\bar{P}|\cdot|H| = \{5\cdot 2,\ 5\cdot 3,\ 5\cdot 4\}$$

$$= \{10,\ 15,\ 20\} \quad ↯$$

↑ Normal subs of this order don't exist by lemma 1.

Complete the proof shown $|H| \neq 6, 12$

Pf $|H| = 6 \xRightarrow{\text{HW 6}} H \simeq \mathbb{Z}_6$ or S_6

so $\exists\ P \trianglelefteq H$
↑ order 3

Lemma * $\longrightarrow$ so $P \trianglelefteq G$ order 3

But eliminated this.

$|H| = 12 \leadsto \exists\ P \trianglelefteq H$ order 3 or 4

Lemma * $\quad P \trianglelefteq G$ " " "

But eliminated ↯

Note If $|G| = 60$ & G is simple

$$\Longrightarrow G \simeq A_5$$

Idea

① Show $n_2 = 5$

Let $P = Syl_2(G)$ $N = N_G(P)$.

$$|G/N| = n_2 = 5$$

$G \circlearrowleft G/N$ left mult get

$$G \longrightarrow S_5$$

② Injective

③ Image is precisely A_5.

Know A_5 simple.

Goal

Theorem A_n simple $\forall$ $n \geq 5$

Use induction on n.

$G = A_n \circlearrowleft \{1, \dots, n\}$

Lemma 1

$G_i \simeq A_{n-1}$ (so simple by induction)

Pf/ G_i is even permutations of

$$\{1, 2, \dots, i-1, i+1, \dots, n\}$$

Lemma 2

Let $H \trianglelefteq A_n$ proper.

1) $\tau_1, \tau_2 \in H$. If $\tau_1(i) = \tau_2(i)$
then $\tau_1 = \tau_2$

2) If $\tau \in H$ & $\tau(i) = i$ then
$\tau = 1$.

Pf/ Prove 2). 1) follows letting $\tau = \tau_1 \tau_2^{-1}$

$$H \trianglelefteq G \implies \underset{\substack{\rotatebox{90}{$\in$} \\ \tau \leftarrow}}{H \cap G_i} \trianglelefteq G_i$$

So G_i simple $\implies H \cap G_i = G_i$

i.e. $G_i \subseteq H$.

Exercise $\sigma \in G$

$$\sigma G_i \sigma^{-1} = G_{\sigma(i)}$$

Therefore $\sigma G_i \sigma^{-1} = G_{\sigma(i)} \subseteq H$

(H normal)

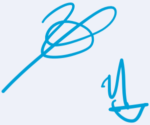
$\Longrightarrow (\underline{G_1, G_2, \ldots, G_n}) \leq H.$

Claim $\langle G_1, \ldots, G_n \rangle = A_n.$

A_n is generate by $(i_1\ i_2)(i_3\ i_4) \ni \sigma$

But $n \geq 5$, so $\exists$ $j \neq i_1, i_2, i_3, i_4$

Then $\sigma \in G_j$



Thus $H = A_n$

Lemma 3

$\tau \in H \trianglelefteq G$ & nontrivial. $\tau \neq 1.$

Then τ is a product of disjoint 2-cycles.

Pf/ $\tau = (a_1\ a_2\ a_3 \ldots)(b_1\ b_2\ \ldots)\ldots$

disjoint.

$n \geq 5$, so $\exists$ $(a_3\ x\ y) = \sigma \in A_n$

w/ $x, y \neq a_1, a_2, a_3$

$x \neq y.$

$\in H$

$\tau_1 = \sigma\tau\sigma^{-1} = (a_1\ \ a_2\ \ \sigma(a_3))(\sigma(b_1)\ \sigma(b_2)\dots)$

① $\tau_1 \neq \tau$ $\quad (\tau(a_2) \neq \tau_1(a_2))$

② $\tau_1(a_1) = a_2 = \tau(a_1)$.

Contradicts Lemma 2

Know $\tau \in H$ & $\tau \neq 1$ $\quad n \geq 5$ by Lemma 2

$$\tau = (a_1\ \ a_2)(a_3\ \ a_4)(a_5\ \ a_6)\dots$$

disjoint.

$$\sigma = (a_1\ a_2)(a_3\ \ a_5) \in A_n$$

$$\tau_1 = \sigma\tau\sigma^{-1} = (a_1\ \ a_2)(a_5\ \ a_3)(a_3\ \ a_3)\dots \in H$$

① $\tau \neq \tau_1$

② $\tau(a_1) = \tau_1(a_1)$

↯ contradicts Lemma 1

Direct & Semidirects Products

Def^{n}

$G_1, G_2, \dots, G_n$ groups. Their direct product is

factors

$$G_1 \times G_2 \times \cdots \times G_n = \{(g_1, g_2, \dots, g_n) \mid g_i \in G_i\}$$

$$\begin{aligned}(g_1, g_2, \dots, g_n)(h_1, h_2, \dots, h_n) \\ = (g_1 h_1, g_2 h_2, \dots, g_n h_n)\end{aligned}$$

Prop

This is a group of order is

$$|G_1| \cdot |G_2| \cdot \dots \cdot |G_n|.$$

Pf/ Exercise.

Example

$$\mathbb{Z}_3 \times S_3 \times GL_2(\mathbb{F}_3) = \left\{\left(x^i, \sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right\}$$

$$\begin{aligned}\left(x^i, \sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot \left(x^j, \sigma', \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \\ = \left(x^{i+j}, \sigma\sigma', \left(\cdots \right)\right)\end{aligned}$$

Important maps & identifications

1) $$\begin{aligned}\iota_j : G_j &\longrightarrow G_1 \times \cdots \times G_j \times \cdots \times G_n \\ g &\longmapsto (1, 1, \dots, g, \dots, 1)\end{aligned}$$

j'th position.

Exercise i_j is an injective homomorphism.

We write $G_j \leq G_1 \times \cdots \times G_j \times \cdots \times G_n$

Prop $G_j \trianglelefteq G_1 \times \cdots \times G_n$.

Pf/

$$G_1 \times \cdots \times G_n \longrightarrow G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_n$$

$$(g_1, \ldots, g_n) \longmapsto (g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n)$$

Surjective homomorphism w/ $\ker = G_j$

So $G_j \trianglelefteq G_1 \times \cdots \times G_n$.

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2) **Projection**

$$\pi_j : G_1 \times \cdots \times G_n \longrightarrow G_j$$

$$(g_1, \ldots, g_j, \ldots, g_n) \longmapsto g_j$$

Exercise This is a surjective hom w/

$\ker = G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_n$

Upshot G_j is a subgroup & a quotient of $G_1 \times \cdots \times G_n$.

Moral H is a normal subgroup & quotient of G

maybe $G = H \times K$?

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i'th coord & j'th coord don't talk.

$G_i, G_j \trianglelefteq G_1 \times \dots \times G_n$ $\quad i \neq j$

$x \in G_i$ $\quad y \in G_j$

Then $xy = yx$

Pf/ $xy = (1 \dots x, \dots, y \dots 1) = yx$

$i < j$