Feb4

Thursday, February 4, 2021 12:49 PM

Question G a group.

$x \in G$ $a, b \in \mathbb{Z}$.

does $x^a x^b = x^{a+b}$?

Pf/ $a, b > 0$

$$x^a \cdot x^b = (\underbrace{x \cdot x \cdots x}_{a})(\underbrace{x \cdots x}_{b}) = \underbrace{x \cdots x}_{a+b} = x^{a+b}$$

$\emptyset$

General case as an exercise.

$$x^a x^{-b} = (\underbrace{x \cdots x}_{a})(\underbrace{x^{-1} \cdots x^{-1}}_{b}) = x^{a-b}$$

$\emptyset$.

Recall:

* In linear algebra you studied functions

$$F: V \longrightarrow W$$

btwn vector space w/ F linear.

⇝ F represented by Matrix

⇝ Respects + & scalar mult.

* Calculus studied functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

f, cts or differentiable.

we introduced same notion for groups:

Defn G, H groups, A group homomorphism is a function

$$\phi : G \longrightarrow H$$

s.t. $\forall\ x, y \in G$

$$\phi(xy) = \phi(x)\phi(y)$$

If ϕ is also bijective, then it is an isomorphism.

Rmk

Isomorphic groups are "the same" in an abstract sense.

If $G \cong H$, then a group theoretic property is true for $G \Longleftrightarrow$ true for H.

Ex V, W be 2 vector spaces.

Groups under +.

$F : V \longrightarrow W$ a linear map.

$\Longrightarrow F(v_1 + v_2) = F(v_1) + F(v_2)$

So it is a homomorphim.

If $\det F \neq 0 \Rightarrow F$ is bijectia

So an iso.

Example of iso

Let A, B be 2 sets, & $\underline{\theta : A \longrightarrow B}$ a bijection.

HW2 #5 $\Rightarrow$ $S_A = \{ f : A \to A \mid f \text{ bijective} \}$

groups $S_B = \{ g : B \to B \mid g \text{ bijective} \}$

Then $S_A \cong S_B$ ←

Proof I'm going to tell you the map

$$\phi : S_{\underline{A}} \xrightarrow{\simeq} S_B$$

Input: permutation of A. $f : A \longrightarrow A$

Output: permutation of B $\phi(f) : B \longrightarrow B$ ←

$\theta\left(f\left(\theta^{-1}(b)\right)\right)$

$$\begin{tikzcd}
A \arrow[r, "f"] & A \arrow[d, "\theta"] \\
B \arrow[u, "\theta^{-1}"] \arrow[r, dashed, "\phi(f)"] & B
\end{tikzcd}$$

$$\phi(f) = \theta \circ f \circ \theta^{-1}$$

Exercise

① $\phi(\sigma)$ is bijective.

② ϕ is a homomorphism.

③ ϕ is bijective.

Remarks

① Look familiar.

$F: V \longrightarrow V$ M change of basis.

To compute F w/ new basis

$$M \circ F \circ M^{-1}$$

② Let A be any finite set $|A| = n$.

$\Longrightarrow$ $S_A \cong S_n$ symmetric group.

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§ Group Actions

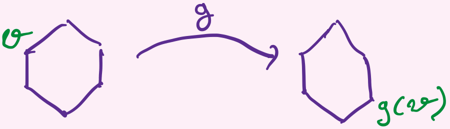
Powerful tool to define & study structure of groups.

Example

D_{12} acts on the hexagon (or its vertices)

An element $g \in D_{12}$ does stuff to the hexagon.

↑ acts on



g is an elt of gp... but also a bijective function on this object.

Tautologically, this respect group structure

$$g_1 \cdot g_2 \cdot (v) = (g_1 g_2)(v)$$

b/c mult is composition.

Example 2

S_n defined by action on $\{1, \dots, n\}$

Defn (Group Actions)

Let G be a group and A a set. A group action of G on A is a function

$$\begin{aligned} G \times A &\longrightarrow A \\ (g, a) &\longmapsto g\cdot a \qquad (ga) \end{aligned}$$

Satisfying

→ (i) $g_1\cdot(g_2\cdot a) = (g_1 g_2)\cdot a \qquad \forall\ g_1, g_2 \in G, \quad \forall\ a \in A$

→ (ii) $1\cdot a = a \qquad \forall\ a \in A.$

Notation G acts on A $\longleftrightarrow$ $G \circlearrowright A$

Think about this as saying that each element of G gives a "symmetry" of A or a permutation of A.

This can be made precise:

Defn/Propn

Fix an action $G \times A \longrightarrow A$.

For $g \in G$ define the "permutation associated to g" to be

$$\begin{aligned} \sigma_g : A &\longrightarrow A \\ a &\longmapsto g\cdot a \end{aligned}$$

Claim 1: σ_g is bijective

Proof

Construct an inverse: $\sigma_{g^{-1}}$

Show $\underbrace{\sigma_g \circ \sigma_{g^{-1}} = id_A}$ & $\sigma_{g^{-1}} \circ \sigma_g = id_A$

$$
\begin{aligned}
(\sigma_g \circ \sigma_{g^{-1}})(a) &= \sigma_g(\sigma_{g^{-1}}(a)) \\
&= g\cdot(g^{-1}\cdot a) \\
(1) \quad &= (gg^{-1})\cdot a \\
&= 1\cdot a \\
(11) \quad &= a
\end{aligned}
$$

$\emptyset$

Described a process

$$\begin{tikzcd} & & & & \text{group} \arrow[dll] \arrow[d] \\ (G\times A \arrow[r] & A) \arrow[r, squiggly] & (G \arrow[rr, "\phi"] & & S_A) \\ & & g \arrow[rr, mapsto] & & \sigma_g \end{tikzcd}$$

Claim 2 $\phi : G \longrightarrow \underline{S_A}$ is a homomorphism.

pf/ $x, y \in G$.

$\phi(xy) \longleftrightarrow \underline{\sigma_{xy} : a \mapsto (xy)\cdot a}$

$\phi(x)\phi(y) \longleftrightarrow \sigma_x \underset{\uparrow}{\circ} \sigma_y$

$\sigma_x(\sigma_y(a)) = x\cdot(y\cdot a)$ (def)

$= (xy)\cdot a$ (1)

$= \sigma_{xg}(a)$

The homomorphism
$$\begin{aligned} G &\longrightarrow S_A \\ g &\longmapsto \sigma_g \end{aligned}$$
is called the permutation representation of the action $G \times A \longrightarrow A$

Remark

$$\left\{ \begin{matrix} \text{Groups actions} \\ G \circlearrowright A \end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix} \text{Homomorphism} \\ G \longrightarrow S_A \end{matrix} \right\}$$

Given $\phi : G \longrightarrow S_A$

Define $G \times A \longrightarrow A$

$$g \cdot a = \phi(g)(a)$$

Check This is an action & these constructions are inverses.

Def^n

Let $G \circlearrowright A$ $\quad (G \times A \longrightarrow A)$,

1) The kernel of the action is the set of elements of G acting trivially

$$\{ g \in G \mid g \cdot a = a \text{ all } a \in A \} \geq 1$$

2) If only 1 acts trivially (kenel $= \{1\}$) then we call the action faithful.

↑ equiv perm rep is injective

Examples

1) Trivial action of G on A.

$$g \cdot a = a \quad \text{all } g \in G, \ a \in A.$$

Kernel $= G$.

Permutation Rep

$$\phi : G \longrightarrow S_A$$
$$g \longmapsto \mathrm{id}_A = 1$$

2) Let V be a n dim'l $\mathbb{R}$-vector space.

$(\mathbb{R}\setminus\{0\}, \times) = \mathbb{R}^\times \circlearrowright V$ by scalar mult.

$$\lambda \cdot v = \lambda v$$
$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

$1 \cdot v = v$
$\tau \cdot \lambda \cdot v$
$= (\tau\lambda) v$

Faithful If $\lambda v = v \ \forall v$
$\Rightarrow \lambda = 1.$

$G = GL_n(\mathbb{R}) = \{\text{invertible } n \times n \text{ matrices}\}$

$G \circlearrowright V$

$$M \cdot v = M v.$$

perm rep
$M \longmapsto$ Linear map
$V \xrightarrow{M} V$

Exercise Faithful.

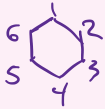
3) A nonempty set

$$\begin{array}{ccc} S_A \times A & \longrightarrow & A \\ (f, a) & \longmapsto & f(a) \end{array}$$

Perm rep $\quad \phi: S_A \xrightarrow{id} S_A$

$$f \longmapsto f$$

4) $\{1, 2, 3, 4, 5, 6\} = A$



$$\begin{array}{ccc} D_{12} \times A & \longrightarrow & A \\ (\alpha, v) & \longmapsto & \alpha(v) \end{array}$$

← where does vertex go under the symmetry

Get $\quad D_{12} \longrightarrow S_A \cong S_6$

Observe Get $\quad D_{2n} \longrightarrow S_n \quad \forall\, n \geq 2$

same way.

5) Groups act on themselves in various ways.

Left Mult $\quad G \circlearrowleft G$

$$g \cdot a = ga$$

Faithful

$ga = a$

$\Rightarrow g = 1$

upshot

Get a permutation representation.

$$G \xrightarrow{\phi} S_G \simeq S_n \leftarrow$$

↑ any weird group

↑ G finite.

↑ we understand in terms of cycle decoms etc.

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§ Subgroups

Observation:

"Looks like there is a copy of $\mathbb{Z}/n\mathbb{Z}$ living inside D_{2n}"

1) R = set of rotations

↑ $= \{1, r, r^2, r^3, \dots, r^{n-1}\} \subseteq D_{2n}$

i) The product of 2 rotations is a rotation. ←

$$r^i \cdot r^j = r^{i+j} \in R \leftarrow$$

ii) The inverse of a rotation is a rotation.

$$(r^i)^{-1} = r^{n-i} \in R \leftarrow$$

iii) The identity is a rotation. ←

R is a group living inside D_{2n}.

2) $$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\simeq\ \phi} R$$

$\bar{a} \longmapsto r^a$

i) Well defined

If $a \equiv b \mod n \Longleftrightarrow r^a = r^b$

$\uparrow$ HW2 8(c) $\leftarrow$

& $|r| = n$

ii) Inj $r^a = r^b$

$\Longrightarrow \bar{a} = \bar{b}$

iii) $\phi(\bar{a} \underset{\uparrow}{+} \bar{b}) = r^{a+b} = r^a \cdot r^b = \phi(\bar{a}) \underset{\uparrow}{} \phi(\bar{b})$

iv) Surj $r^a = \phi(\bar{a})$ ✓

Def^n

Let G be a group & $H \subseteq G$ a subset. Then H is a subgroup of G if it is closed under mult & inversion.

* $a, b \in H \rightsquigarrow ab \in H$

$\uparrow$ mult in G

* $a \in H \rightsquigarrow a^{-1} \in H$

$\uparrow$ invert as elt of G!

We write $H \leq G$

Lemma

If H is a subgroup of G, then it is a group

Pf/ Mult on G is a binary operation on H by closure under $\times$.

Assoc is inherited from G.

Identity/ Inverse

$x \in H. \implies x^{-1} \in H$ by closure under inverses.

so $1 = x \cdot x^{-1} \in H$ by closure under $\times$.

Examples:

① $R \leq D_{2n}$ (rotations)

$(R \cong \mathbb{Z}/n\mathbb{Z})$

② Any Group comes equipped w/ 2 subgroups.

$G \leq G$ $\{1\} \leq G$

③ Under $+$

$\mathbb{Z} \leq \mathbb{Q}$

$\mathbb{Q} \leq \mathbb{R}$

$2\mathbb{Z} \leq \mathbb{Z}$ $n\mathbb{Z} \leq \mathbb{Z}$

‖ {even ints}   ‖ {mults of n}

④ Non examples.

a) $(\mathbb{Q}\setminus\{0\}, \times) \not\leq (\mathbb{Q}, +)$

Is a subset

Is a group.

Not Some operation.

Moral Subgroup needs to have same operation.

b) $(\mathbb{Z}_{\geq 0}, +) \not\leq (\mathbb{Z}, +)$

Closed under +

Not closed under inverses.

Moral: Not enough to check closure under operation.

Lemma: "Is a subgroup of" is a transitive operation.

$K \subseteq H \subseteq G$ $\quad$ G a group.

If $H \leq G$
and $K \leq H$ $\Longrightarrow K \leq G$

Proof: Exercise.