Feb18

Thursday, February 18, 2021 2:35 PM

* First Takehome Released Tomorrow!
Due Monday 2-22

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* HW 5 released Monday. (Short).

Isomorphism Theorems

Theorem 1 (First Isomorphism Theorem)

Let G, H groups

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & H \\ G/\ker\phi \arrow[ur, dashed, "\bar{\phi}"'] & \end{tikzcd}$$

HW
$\ker\phi \trianglelefteq G$

s.t. $\bar{\phi}(g\ker\phi) = \phi(g)$.

① $\bar{\phi}$ is injective

② $G/\ker\phi \simeq \mathrm{im}\,\phi$ $\leftarrow$ If ϕ surjective $\rightsquigarrow \bar{\phi} : G/\ker\phi \xrightarrow{\sim} H$.

Rmk (Commutative Diagrams)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] \arrow[dr, phantom, bend left, "\circlearrowleft"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

means

$h = g \circ f$

Proved last time

Example (Circle Group)

Define $\|\cdot\| : \overset{(\mathbb{C}\setminus\{0\},\, \times)}{\mathbb{C}^\times} \longrightarrow \mathbb{R}^\times$

$$a+bi \longmapsto \sqrt{a^2+b^2}$$

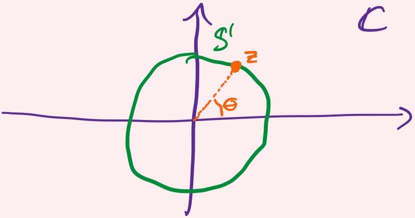
Check $z, w \in \mathbb{C}^\times$

$\Longrightarrow \|zw\| = \|z\| \cdot \|w\|$

So $\|\cdot\|$ is a homomorphism.

Let $S^1 = \ker(\|\cdot\|) = \{z \in \mathbb{C}^\times \mid \|z\| = 1\} \trianglelefteq \mathbb{C}^\times$

Picture



$x^2+y^2=1$

$z = \cos\theta + i\sin\theta \underset{\uparrow}{=} e^{i\theta}$

Taylor series

$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots$

$i\sin\theta = i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right.$

$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + \dots$

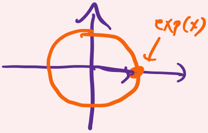
Define

$$\begin{aligned} \exp : \overset{\text{undr } +}{\mathbb{R}} &\longrightarrow S^1 \\ x &\longmapsto e^{2\pi i \cdot x} \end{aligned}$$

$$\exp(x+y) = e^{2\pi i (x+y)} = e^{2\pi i x} \cdot e^{2\pi i y} = \exp(x)\exp(y).$$

$\rightsquigarrow$ exp is a surjective homomorphism.

What is the kernel?



$$\exp(x) = 1 \iff e^{2\pi i x} = 1$$

$$\iff 2\pi x = 2\pi(n) \quad n \in \mathbb{Z}.$$

$$\iff x \in \mathbb{Z}.$$

$$\rightsquigarrow \ker(\exp) = \underline{\underline{\mathbb{Z}}}.$$

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d] & S^1 \\ \mathbb{R}/\mathbb{Z} \arrow[ur, "\sim"'] & \end{tikzcd}$$

Thus $\underline{\mathbb{R}/\mathbb{Z} \simeq S^1}$ First Iso Thm

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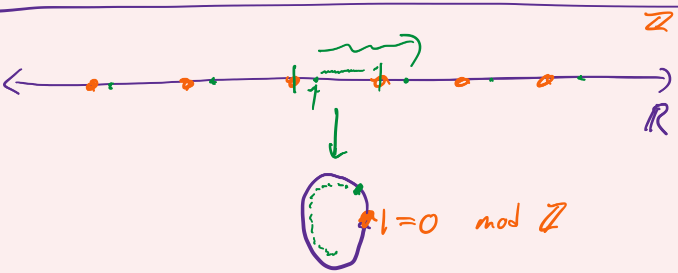


$1 = 0 \mod \mathbb{Z}$

Exercise:

Use this setup to show

$$\begin{array}{ccl} (\mathbb{C}^\times)^{\text{tors}} & = & \{ z \mid z^n = 1 \text{ some } n>0 \} \\ \wr\| & & = \mu \ni i \\ \mathbb{Q}/\mathbb{Z} & & \end{array}$$

Next one will concern the following

Def $H, K \le G$. Form a set

$$HK = \{ hk \mid h \in H, k \in K \}$$

Prop (*)

$$|HK| = \frac{|H|\cdot|K|}{|H\cap K|} \longleftarrow$$

Proof

$$HK = \bigcup_{h\in H} \underset{\uparrow}{hK} \longleftarrow$$

show disjoint (HW 4 partition)

All have $|K|$ ← exercise

$$\Longrightarrow |HK| = \overset{\#\ \text{distinct cosets}}{n}\,|K|$$

Compute $h_1, h_2 \in H$

$$h_1K = h_2K \iff h_2^{-1}h_1 \in K \cap H$$

$$\Longleftrightarrow h_1(K\cap H) = h_2(K\cap H)$$

$$\overset{n}{\#\begin{pmatrix}\text{distinct } hK \\ h\in H\end{pmatrix}} = \#\begin{pmatrix}\text{distinct} \\ h(K\cap H) \\ h\in H\end{pmatrix} = \left|H/(K\cap H)\right|$$

$$\xrightarrow{\text{Lagrange}} = \frac{|H|}{|K\cap H|} = n$$

$$|HK| = n\cdot|K| = \frac{|H||K|}{|K\cap H|} \qquad \varnothing$$

Notice we never use $HK \le G$.

This will fail.

Ex $G = S_3$ $\quad H = \langle (1\,2)\rangle$, $K = \langle (2\,3)\rangle$ $\;\Big|\;$ $H\cap K = \{(1)\}$

$$|HK| = \frac{|H|\cdot|K|}{|H\cap K|} = \frac{2\cdot 2}{1} = 4$$

$4 \nmid 6$ so $HK \not\le G$. by Lagrange

Q When is $HK \le G$?

Prop $HK \le G \Longleftrightarrow \underline{HK = KH}$

Proof

$\Leftarrow$ h_1k_1 & h_2k_2 in HK

$$(h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK \quad ✓$$

$$h_1\underbrace{k_1h_2}_{\in KH = HK}k_2 = h_1\hat{h}_2\hat{k}_1k_2 \in HK \quad ✓$$

$$\Longrightarrow H, K \overset{\text{HW 1(c)}}{\le} HK.$$

$$\rightsquigarrow KH \subseteq HK$$

If $hk \in HK \rightsquigarrow hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$

$$HK \subseteq KH \quad ∅$$

Corollary $H, K \le G$. $H \le N_G(K)$ (e.g. if $K \trianglelefteq G$)

Then $HK \le G$.

Pf/ $h \in H$

$\rightsquigarrow hKh^{-1} = K$ So $hK = Kh$

$$\xRightarrow{\forall h \in H} HK = KH \quad ∅$$

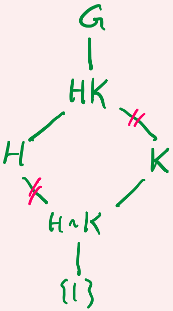
Theorem 2 (Second Isomorphism Theorem).

(Diamond Iso Thm).

$HK \le G$

G a group. $H, K \leq G$. $\underline{H \leq N_G(K)}$

(e.g. $K \trianglelefteq G$)



Then

① $K \trianglelefteq HK$ ← ✓

② $H \cap K \trianglelefteq H$ ✓

③ $\frac{HK}{K} \simeq \frac{H}{H \cap K}$ ← ✓

Proof

$H \leq N_G(K)$ assumption

$K \leq N_G(K)$ HW

$\Longrightarrow HK \leq N_G(K)$

So $x \in HK \rightsquigarrow xKx^{-1} = K$

$\Longrightarrow K \trianglelefteq HK$ proves ①.

$$\begin{tikzcd}
HK \arrow[rr, "x \mapsto xK"] & & \frac{HK}{K} \\
H \arrow[u, phantom, sloped, "\subseteq"] \arrow[urr, "\phi"] \arrow[d] & & \\
H/H\cap K \arrow[uurr, "\simeq"'] & &
\end{tikzcd}$$

$\phi(h) = hK$.

First iso thm.

Well defined homom b/c restriction of nat'l projection

1) Surjective $hkK = hK = \phi(h)$ ✓

2) $h \in \ker\phi$. $\iff$ $hK = 1K \iff h \in K$

$\rightsquigarrow \ker\phi = H \cap K$ ✓

Rmk

① Special case where $HK = G$ & $H \cap K = 1$

$\Rightarrow G/K = H$

② Reproves prop (*) from earlier.

Lagrange $\dfrac{|HK|}{|K|} = \dfrac{|H|}{|H \cap K|}$

Mult by $|K|$ regain (*)

in case wher $H \leq N_G(K)$.

Theorem 3 (Third isomorphism theorem)

(Cancellation)

$H, K \trianglelefteq G$. $H \leq K$

✓ ① $H \trianglelefteq K$

② $K/H \trianglelefteq G/H$ ✓

③ $\dfrac{(G/H)}{(K/H)} \simeq G/K$. ✓

Proof

① is obvious. $x \in K \leq G$ & $xHx^{-1} = H$

$$\begin{array}{ccc} K/H & \leq & G/H \\ \rotatebox{90}{$\in$} & & \\ \underline{\underline{kH}} & \in & G/H \end{array}$$

$k \in K$, $H \leq K$

$kH \subseteq K$

"kH as a coset in K is $=$ kH as a coset in G"

$$\rightarrow \begin{array}{ccc} K/H & \leq & G/H \\ \shortparallel & & \shortparallel \end{array}$$

$$\rightarrow \{gH \mid g \in K\} \subseteq \{gH \mid g \in G\}$$

$$\begin{aligned} \phi : G/H &\longrightarrow G/K \\ gH &\longmapsto gK \end{aligned}$$

Well defined

$g' \in gH. \implies g' = gh$ $\quad \boxed{h \in H \leq K}$

$\implies g' \in gK$

$\implies g'K = gK$ ✓

Homom Easy. ✓

Surjective $gK = \phi(gH)$ ✓

$$\ker\phi = \{ gH \in G/H \mid \underline{gK = 1K} \}$$

$$= \{ gH \mid g \in K \}$$

$$= K/H$$

$$\begin{tikzcd}
G/H \arrow[r, "\phi"] \arrow[d] & G/K \\
\frac{G/H}{G/K} \arrow[ur, "\overline{\phi}"] &
\end{tikzcd}$$

isom by first
iso thm.

Example 5|15

$$15\mathbb{Z} \le 5\mathbb{Z} \le \mathbb{Z}.$$

$$\begin{array}{ccc}
\mathbb{Z}/15\mathbb{Z} & \supseteq & 5\mathbb{Z}/15\mathbb{Z} \\
\| & & \| \\
\{\overline{0}, \overline{1}, \overline{2}, \dots\} & \supseteq & \{\overline{5}, \overline{10}, \overline{15}\}
\end{array}$$

$$\frac{\mathbb{Z}/15\mathbb{Z}}{5\mathbb{Z}/15\mathbb{Z}} \cong \mathbb{Z}/5\mathbb{Z}$$

Thm 3

Work this out for m|n

Observation

In previous theorem we implicitly used a dictionary

$$H \le \underset{*}{K} \le G \qquad \leftrightsquigarrow \qquad \underbrace{1 \le \underset{*}{K/H} \le G/H}_{\text{subgroup in } G/H.}$$

subgroup btwn H & G

Saw info about K & K/H lined up.

$$K \trianglelefteq G \rightsquigarrow K/H \trianglelefteq G/H$$

Slogan

$$\left(\begin{array}{l}\text{Stuff between}\\ H \;\&\; G\end{array}\right) \longleftrightarrow \left(\begin{array}{l}\text{stuff in}\\ G/H\end{array}\right)$$

Formalize this

Theorem 2/ (Fourth isomorphism theorem)

Lattice iso thm.

Let G be a group. Let $N \trianglelefteq G$ normal.

Then there is a bijection.

$$\left\{\begin{array}{l}\text{Subgroups of } G\\ \text{containing } N\\ N \le H \le G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Subgroups}\\ \text{of } G/N\end{array}\right\}$$

$$H \longmapsto \overline{H} = \frac{H}{N} \leq \frac{G}{N}$$

$$\pi^{-1}(\overline{K}) \longleftarrow\!\!\!\shortmid\ \overline{K}$$

$\uparrow$ $\pi : G \longrightarrow G/N$

$$\pi^{-1}(\overline{K}) = \{ g \in G \mid \pi(g) \in \overline{K} \}$$

$= \text{preimage of } \overline{K}.$

Better yet. For $H, K \leq G$ w/ $N \leq H, K \leq G$. we have

① $H \leq K \iff \overline{H} \leq \overline{K}$

② If $H \leq K$ $\quad |K : H| = |\overline{K} : \overline{H}|$

③ $\overline{\langle H, K \rangle} = \langle \overline{H}, \overline{K} \rangle$

④ $\overline{H \cap K} = \overline{H} \cap \overline{K}$

⑤ $H \trianglelefteq K \iff \overline{H} \trianglelefteq \overline{K}$ ←

Proof: Exercise.

Remark

Break down the study of G into 2 peices.

$H \trianglelefteq G$

$\uparrow$

study H $\qquad$ study G/H

Gain a lot of info about G.

Example: $D_8 = \langle r, s \rangle$.

nontriv Subs of D_8 order 2, 4 (Lagrange)

$R = \langle r \rangle$,

$\langle s, r^2 \rangle = \{s, r^2, sr^2, 1\}$

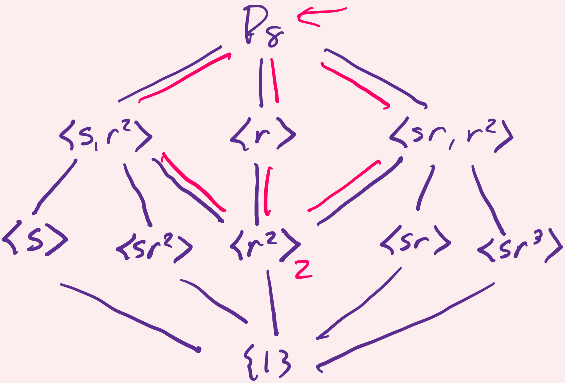
$\langle sr, r^2 \rangle = \{sr, r^2, sr^3, 1\}$

} All ordr 4 subs.

Think about

Subs of ordr 2

$\langle sr^i \rangle \leftarrow \forall i$ $\langle r^2 \rangle$. } All one generator. These are elts of order 2.



$V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 =$ $a = (1,0)$
$b = (0,1)$

$= \langle a, b \mid ab = ba \quad a^2 = b^2 = 1 \rangle$

$$\begin{tikzcd}
 & V_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle a \rangle \arrow[dr, dash] & \langle ab \rangle \arrow[d, dash] & \langle b \rangle \arrow[dl, dash] \\
 & \{1\}. &
\end{tikzcd}$$

$D_8/\langle r^2 \rangle = \underline{\underline{D_8/Z}}$ $(\simeq V_4)$.

4th iso thm

Subs $D_8/Z \longleftrightarrow$ Subs D_8 containing Z.

$$\begin{tikzcd}
 & D_8/Z \leftarrow \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\overline{\langle sr, r^2 \rangle} \arrow[dr, dash] & \overline{\langle r \rangle} \arrow[d, dash] & \overline{\langle sr^3, r^2 \rangle} \arrow[dl, dash] \\
 & \{1\} &
\end{tikzcd}$$

Slogan

Lattice of quotient lives inside lattice of the entire group.

* $H \trianglelefteq G$.

1) Lattice of H is part of latice of G below H.

2) Latice of G/H is part of lattice of G above H.

Observe V_4 abelian All subgroups of D_8/Z are normal

$\Longrightarrow$ All $H \leq D_8$ s.t. $\langle r^2 \rangle \leq H \leq D_8$ are normal.

Arguing same way.

Corollary

Let $N \trianglelefteq G$. Suppose G/N is abelian. If $N \leq H \leq G$. Then $H \trianglelefteq G$.