Feb16

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Let $\phi: G \longrightarrow H$ be a surjective hom.

Let K kernel. Studying 3 groups.

① H

② $\mathrm{Fib}(\phi) = \{\phi^{-1}(h) \mid h \in H\}$

③ $G/K = \{\text{Equiv classes mod } K\}$

$$= \{g \cdot K \mid g \in G\}$$

Proved

Ⓐ
$$\begin{array}{ccc} \mathrm{Fib}(\phi) & \xrightarrow{\sim} & H \\ \phi^{-1}(h) & \longmapsto & h. \end{array}$$

Ⓑ $\underline{g \equiv g'} \bmod K \iff \underline{\phi(g) = \phi(g')}$ $\longleftarrow$

$g = g'k$ some
$\uparrow$ $k \in K.$

Part Ⓑ $\Longrightarrow$ get a bijection

$$\begin{array}{ccc} G/K & \xrightarrow{\gamma} & \mathrm{Fib}(\phi) \simeq H \\ g \cdot K & \longmapsto & \phi^{-1}(\phi(g)) \end{array}$$

$$g\cdot K \longleftarrow\!\!\!| \ \phi^{-1}(h)$$

any $g\in\phi^{-1}(H)$.

Multiplication in G/K $=\{g\cdot K \mid g\in K\}$

Prop The binary operation

$$(g_1\cdot K)(g_2K) = g_1g_2K \longleftarrow$$

makes G/K into a group w/ identity K. $= 1K$

This makes γ an isomorphism.

Pf/ Well defined

If $\hat{g}_1\in g_1K$ & $\hat{g}_2\in g_2K$

Show $\hat{g}_1\hat{g}_2K = g_1g_2K$.

i.e. $\hat{g}_1\hat{g}_2 \equiv g_1g_2 \mod K$.

Know $\hat{g}_1 = g_1k_1$ $\quad \hat{g}_2 = g_2k_2$

So $\hat{g}_1\hat{g}_2 = \underbrace{g_1k_1g_2k_2}$ - - - - -

Ⓑ Apply ϕ

$$\phi(\hat{g}_1 \hat{g}_2) = \phi(g_1 k_1 g_2 k_2)$$

$$= \phi(g_1)\cancel{\phi(k_1)}\phi(g_2)\cancel{\phi(k_2)}, \quad k_i \in \ker\phi \ ✓$$

$$= \phi(g_1 g_2) \quad \text{we win}$$

I inherit groupyness from G (HW 4 8(c))

γ iso homom b/c

$$g\cdot K \xmapsto{\gamma} \phi^{-1}(\phi(g))$$

$$\gamma(g_1 g_2 K) = \phi^{-1}(\phi(g_1 g_2))$$

$$= \phi^{-1}(\phi(g_1)\phi(g_2))$$

$$= \phi^{-1}(\phi(g_1))\phi^{-1}(\phi(g_2))$$

$$= \gamma(g_1 K)\gamma(g_2 K)$$

Rmk

This is probably not the best proof. B/C relied on ϕ to prove well defined. You will give the better proof in 8(b) & 8(c) using K normal instead of that K is a kernel.

Theorem (First isomorphism theorem). Let $\phi : G \longrightarrow H$ be a surjective homomorphism. $K = \ker \phi$.

$$\begin{tikzcd} G/K \arrow[r] & H \\ gK \arrow[r, mapsto] & \phi(g) \end{tikzcd}$$

is a well defined isomorphism.

Proof

$$\begin{tikzcd} G/K \arrow[r, "\sim"] & Fib(\phi) \arrow[r, "\sim"] & H \\ gK \arrow[r, mapsto] & \phi^{-1}(\phi(g)) \arrow[r, mapsto] & \phi(g). \end{tikzcd}$$

Cor Let $\phi : G \longrightarrow H$ be a homomorphism w/ $\ker \phi = K$.

Then $G/K \simeq \operatorname{im} \phi \leq H$.

Pf/ $\phi : G \longrightarrow \operatorname{im} \phi$ surjective. Us F.I.T.

Examples:

① $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}_n = \langle x \mid x^n = 1 \rangle$

$a \longmapsto x^a$

$\ker \phi = \{ a \mid x^a = 1 \}$

$= n\mathbb{Z}$

HW 2 # 7 or 8

$x^a = 1 \iff n \mid a$

$\Longrightarrow$ F.I.T. $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n.$

② $id: G \longrightarrow G.$ $\ker id = \{1\}$

cosets $\underline{g\cdot 1}$

$\Longrightarrow G/\{1\} \simeq G.$

③ $e: G \longrightarrow \{1\}$

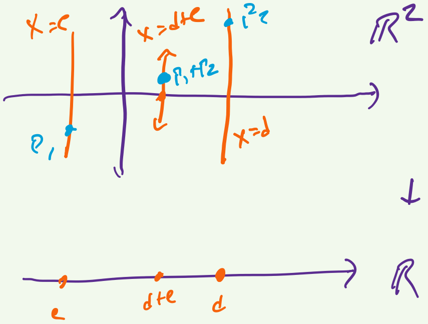
$\ker(e) = G$

$\Longrightarrow G/G \simeq \{1\}.$

$\underline{g\cdot G = G}$

④ $$\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}$$
$$(x,y) \longmapsto x$$

$\ker\pi$
$= \{(0,y)\}$
$=$ y-axis.



$\mathbb{R}^2/(\text{y-axis}) \simeq \mathbb{R}.$

$(x=d) + (x=e)$

$= (x=d+e).$

$\underset{P_1}{(d,y_1)} + (e,y_2)\,P_2$

$= (d+e, y_1+y_2)\,P_3$

$\in x=d+e$

⑤ $$D_{2n} \xrightarrow{\phi} \mathbb{Z}_2 = \langle x \mid x^2 = 1 \rangle$$

$$\| $$

$$\{1, x\}$$

$$sr^i \longmapsto x$$

$$r^i \longmapsto 1$$

Check this is a homomorphism.

$$\ker \phi = R = \{1, r, r^2, \dots, r^{n-1}\}.$$

So $\Longrightarrow$ $D_{2n} / R \simeq \mathbb{Z}_2$

Made all these definitions assuming K was the kernel of some ϕ. Do this w/o that.

Defn $H \leq G$. Define $x \equiv y \mod H$ if $x = yh$ some $h \in H$.

$$G/H := (\text{equivalence classes}) = \text{left cosets}$$

$$= \{ g \cdot H \mid h \in H \}$$

$\uparrow$ $g' \equiv g \iff g' = gh$

Question: When is G/H a group

under $(g_1H)(g_2H) = g_1g_2H$?

Remark

① * Only thing that can go wrong is if this isn't well defined.

② * It is iff $H = \ker \phi$.

Prop ② is an if and only if statement.

If G/H is a group, then $H = \ker \pi$ some homom π.

Pf/ $\pi : G \longrightarrow G/H$ natural projection

$g \longmapsto gH$

$$\pi(g) = 1H$$

$$\Longleftrightarrow \quad g = 1 \cdot h \quad \text{some } h \in H$$

$$\Longleftrightarrow \quad g \in H$$

So $\ker \pi = H$.

G/H group $\Longleftrightarrow$ $H = \ker$ some map

Questions

① Is every subgroup the kernel of

some map? No!

② What properties characterize being in the kernel?

HW # 7 & 8 answer this question.

Show $K = \ker \phi$.

$$N_G(K) = G.$$

i.e. $\forall\ g \in G,\ k \in K$ $\quad gkg^{-1} \in K$.

HW # 7

Def^n

- Let $H \leq G$. We say H is normal in G if $$N_G(H) = G.$$
- Equivalently, if $gHg^{-1} = H \quad \forall\ g \in G$.
- We write $H \trianglelefteq G$ if H is normal in G.

Prop (HW # 8)

If $H \trianglelefteq G$ then

1) G/H is a group

2) $\pi : G \longrightarrow G/H$ is surjective w/ kernel H

Only thing we haven't done at all is show G/H a group. i.e.

$$g_1H \cdot g_2H = g_1g_2H$$ is well defined.

USE NORMALITY

Summarize

This gives a completely intrinsic definition of G/H for any $H \trianglelefteq G$.

WARNING

I could have defined $x \equiv g \mod H$ as $x = hg$ some $h \in H$.

Equivalence classes are $Hg = \{hg \mid h \in H\}$

This is NOT the same.

$$Hg \neq gH$$

Rmk $\{Hg \mid g \in G\} \neq \{gH \mid g \in G\}$

Defn The right cosets of H are the sets $Hg = \{h \cdot g \mid h \in H\}$

The set of right cosets is

$$H\backslash G = \{ Hg \mid g \in G \}$$

Rmk $H\backslash G \neq G/H$ in general. ←

Prop $H \leq G$. Then

$$H \trianglelefteq G \iff gH = Hg \quad \forall g \in G.$$

Proof $H \trianglelefteq G \iff \forall g \in G,\ gHg^{-1} = H$

$\iff gH = Hg \quad \forall g.$

mult on right by g.

$\varnothing$

$A = \{a_1, \dots, a_n\}$

$B = \{b_1, \dots, b_m\}$ $\leq G.$ $Ag = Bg$

$\iff A = B$

Show

$$\begin{tikzcd}
A \arrow[r] & Ag \\
a \arrow[r, mapsto] & ag \arrow[d, equal] \\
& Bg
\end{tikzcd}$$

bijective

Details as exercise.

Examples

① G an abelian group. $H \leq G$.

Then $H \trianglelefteq G$. | G/H a group.

$$\begin{array}{c} xHx^{-1} = H \\ \rotatebox{90}{$\in$} \\ xhx^{-1} = xx^{-1}h = h \end{array}$$

② $G = \mathbb{Z}$. $\quad n\mathbb{Z} \trianglelefteq \mathbb{Z}$

$\uparrow$ all subgroups.

$$\rightsquigarrow G/H \simeq \mathbb{Z}/n\mathbb{Z}$$

"The quotients of $\mathbb{Z}$ are cyclic"

③ $G \simeq \underset{\underset{\langle x \mid x^n = 1\rangle}{\shortparallel}}{Z_n} \geq H = \langle x^d \rangle$ for some $d \mid n$.

$$G/H = \{1H, xH, x^2H, \dots, x^{d-1}H\} \quad \overset{\overset{H}{\shortparallel}}{x^dH}$$

$$= \langle xH \mid (xH)^d = 1 \rangle$$

$$\cong Z_d.$$

$x^{d+1}H$
$= x^d \cdot H \cdot xH$
$= xH \dots$

Theorem

Quotients of cyclic groups are cyclic.

Ex

④ $\mathbb{Z}/6\mathbb{Z} \geq \langle 2 \rangle = \{0, 2, 4\}$

$(\mathbb{Z}/\underline{6\mathbb{Z}})/(2\mathbb{Z}/\underline{6\mathbb{Z}}) = \{0+\{0,2,4\}, 1+\{0,2,4\}\}$
$\cong \mathbb{Z}/2\mathbb{Z}.$

⑤ $H = \langle (1\,2) \rangle \leq S_3$ $\quad H = \{(1\,2), 1\}$

$(1\,3)H = \{(1\,3), (1\,2\,3)\}$

$(1\,3)(1\,2)$

$H(1\,3) = \{(1\,3), (1\,3\,2)\}$

$(1\,2)(1\,3)$

$\rightsquigarrow H \ntrianglelefteq S_3$ left & right cosets don't agree.

$S_3/H \neq H \backslash S_3$

Exercise write down both & inspect.

Prop

$H \leq Z(G) \leq G.$

Then $H \trianglelefteq G.$

Proof Same as for G abelian.

$Z(G) = \{h \in G \mid hg = gh \ \forall g \in G\}$

$h \in H$. Then $ghg^{-1} = h$.

$\implies gHg^{-1} = H$.

G any group. Always look @ $G/Z(G)$.

Example

$Z = Z(D_8) = \{1, r^2\}$

what is D_8 / Z.

Cosets

→ $Z = \{1, r^2\}$ $\quad sZ = \{s, sr\}$

$rZ = \{r, r^3\}$ $\quad srZ = \{sr, sr^3\}$

↑ This partitions D_8.

$|D_8/Z| = 4$ Not Z_4

$(rZ)^2 = r^2 Z = 1$

$(sZ)^2 = s^2 Z = 1$

$(srZ)^2 = (sr)^2 Z = 1$

↑ All have order 2.

Show later that this implies

$D_8/Z \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Observation

(Size of coset) × (# of cosets) = (size of group).

Putting this together formally we get

Langrage's Theorem

G a finite group. $H \leq G$. Then.

① $|H|$ divides $|G|$.

② $|G/H| = |G| / |H|$

③ $|H \backslash G| = |G| / |H|$.

Proof You!

Defⁿ Let $H \leq G$. The index of H in G is

$$|G:H| = |G/H| \underset{\uparrow}{=} |G| / |H|.$$

If finite

Example

$$|\mathbb{Z} : n\mathbb{Z}| = n.$$

$$|\mathbb{R}^2 : \mathbb{R}| = \infty$$

Impt Aside

G abelian $H \trianglelefteq G$.

H abelian not enough.

$H = \langle (1\ 2) \rangle \leq S_3$

$H \simeq \mathbb{Z}_2$ not normal

① $H = \langle (1\ 2\ 3) \rangle \leq S_3$.

Show $H \trianglelefteq S_3$.

$$H \leq N_{S_3}(H) \leq G.$$

HW 4 #3(c)

3

6

divisible by 3 divide 6.

Lagrange

But $|G:H| = 6/3 = 2$.

So $H = N_{S_3}(H)$ or $N_{S_3}(H) = G$.

If I can exhibit one element of normalizer not in H, we win.

$$(1\ 2)(1\ 2\ 3)(1\ 2)^{-1} = (1\ 2)\underbrace{(1\ 2\ 3)(1\ 2)}$$

$$= (1\ 2)(1\ 3)$$

$$= (1\ 3\ 2) = (1\ 2\ 3)^2$$

so $(1\ 2) \in N_G((1\ 2\ 3))$

$$\Longrightarrow N_{S_3}(H) = S_3$$

$$\Longrightarrow H \trianglelefteq S_3 \qquad \varnothing$$

Theorem Let $H \le G$. & $|G:H| = 2$.

Then $H \trianglelefteq G$.

Pf/ $\underset{\substack{\uparrow \\ n}}{H} \le \underset{\substack{\uparrow \\ n \text{ or } 2n}}{N_G(H)} \le \underset{\substack{\uparrow \\ 2n}}{G}$ (Do it for finite groups)

Pick $g \notin H$. $gHg^{-1} = H$ or gH

(2 cosets H, gH)

If it equals gH then $\exists\, h' \in H$ s.t.

$$ghg^{-1} = g \cdot h'$$

$$hg^{-1} = h'$$

$$g^{-1} = h^{-1}h'$$

$$\longrightarrow g^{-1} \in H \text{ so } g \in H \ \lightning$$

$$\varnothing$$

Nicer proof

$g \notin H$. 2 left cosets

$H \qquad gH$

2 right cosets

$H \qquad Hg$

$\longrightarrow \; gH = Hg$ so $H \trianglelefteq G$

$\uparrow$

$gH = G \setminus H = Hg$ b/c partition.

Observe normality is not transitive.

$$\overset{2}{\langle s \rangle} \underset{\substack{\trianglelefteq \\ \uparrow \\ \text{index } 2}}{\le} \overset{4}{\langle s, r^2 \rangle} \underset{\substack{\trianglelefteq \\ \uparrow \\ \text{index } 2}}{\le} \overset{8}{D_8}$$

But $\langle s \rangle \ntrianglelefteq D_8$