Classification Thm

* Classify all finitely generated abelian grps.

Question: What can $|G|$ tell us?

Ex (So far)

$|G| = p \Rightarrow G \simeq \mathbb{Z}_p$

$|G| = p^2 \Rightarrow G \simeq \begin{cases} \mathbb{Z}_{p^2} \\ \mathbb{Z}_p \times \mathbb{Z}_p \end{cases}$

$|G| = pq$, $p<q$, $p \nmid q-1$ $\Rightarrow$ G abelian, in fact $G \simeq \mathbb{Z}_{pq}$

Main tool: Sylow theorems.

Today Example $|G| = 45 = 3^2 \cdot 5$

Defn Elementary abelian group of order p^n is an abelian group V s.t.

1) $|V| = p^n$
2) $x \in V$ $\quad |x| \leq p$

Exercise $V \simeq \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n\text{-times}}$

$\simeq (\mathbb{Z}/p\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p\mathbb{Z})$

$= \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{Z}/p\mathbb{Z}\}$

$(a_1, \ldots, a_n) + (b_1, \ldots, b_n)$
$= (a_1+b_1, \ldots, a_n+b_n)$

Remark $V \simeq (\mathbb{F}_p)^n$ is a vector space of dim n over $\mathbb{F}_p$.

$e_1 = (1, 0, \ldots, 0)$
$e_2 = (0, 1, \ldots, 0)$
$\vdots$
$e_n = (0, 0, \ldots, 0, 1)$

basis of V

Recall $\psi: W \longrightarrow W'$ linear map of vector spaces of a field F, if

① $\psi(\vec{v} + \vec{w}) = \psi(\vec{v}) + \psi(\vec{w})$ $\forall \vec{v}, \vec{w} \in W$

② $\lambda \in F$, $\vec{v} \in W$ $\quad \psi(\lambda \vec{w}) = \lambda \psi(\vec{w})$

Prop $V \simeq (\mathbb{Z}/p\mathbb{Z})^n$
Let $\phi: V \longrightarrow V$ a function. ϕ is a homomorphism $\iff$ ϕ is linear.

Pf/ ($\Leftarrow$) Cond 1) $\Rightarrow$ ϕ hom

($\Rightarrow$) 1) ✓
2) compatible w/ scalar

$\lambda \in \mathbb{F}_p \rightsquigarrow \lambda = \bar{m}$ some $m \in \{0, \ldots, p-1\}$

$\lambda \cdot \vec{v} = \bar{m}(a_1, \ldots, a_n)$
$= (\overline{m a_1}, \ldots, \overline{m a_n})$
$= (\underbrace{a_1 + \cdots + a_1}_{m}, \ldots, \underbrace{a_n + \cdots + a_n}_{m})$
$= \underbrace{(a_1, \ldots, a_n) + \cdots + (a_1, \ldots, a_n)}_{m\text{-times}}$
$= \vec{v} + \cdots + \vec{v}$

$\phi(\lambda \vec{v}) = \phi(\vec{v} + \cdots + \vec{v})$
$= \phi(\vec{v}) + \cdots + \phi(\vec{v})$
$= \lambda \phi(\vec{v})$.

Recall
$\mathrm{Aut}(V) = \{\phi: V \longrightarrow V \text{ iso}\}$

$GL_n(\mathbb{F}_p) = \{\phi: \mathbb{F}_p^n \longrightarrow \mathbb{F}_p^n \text{ bij. linear}\}$ HW

Corollary V elementary abelian order p^n
$\mathrm{Aut}(V) \simeq GL_n(\mathbb{F}_p)$.

Example
$V_4 = \{1, a, b, c\}$ $\quad a^2 = b^2 = c^2 = 1$

$V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \leftarrow$ element abelian order 2^2

$\mathrm{Aut}(V_4) \simeq GL_2(\mathbb{F}_2)$

Prop $GL_2(\mathbb{F}_2) \simeq S_3$

Pf/ $GL_2(\mathbb{F}_2) \circlearrowright \{a, b, c\}$
ϕ is $\phi: V_4 \xrightarrow{\sim} V_4$

$\phi \cdot a = \phi(a)$
$\phi \cdot b = \phi(b)$
$\phi \cdot c = \phi(c)$

Kernel = id b/c if $\phi(a) = a$, $\phi(b) = b$, $\phi(c) = c$ $\Rightarrow \phi = id_{V_4}$
also $\phi(1) = 1$

Get (perm rep)
$GL_2(\mathbb{F}_2) \longrightarrow S_3$ injective.
Both sides have order 6. So bijective

Recall
$\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$

$\Longrightarrow$ $|\mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})| = p^2 - |\{0, p, 2p, \ldots, (p-1) \cdot p\}|$
$= p^2 - p$

Prop V a group $|V| = p^2$

$\mathrm{Aut}(V) = \begin{cases} (\mathbb{Z}/p^2\mathbb{Z})^\times & : \text{cyclic} \\ GL_2(\mathbb{F}_p) & : \text{elem.} \end{cases}$

$|\mathrm{Aut}(V)| = \begin{cases} p^2 - p & : \text{cyclic} \\ (p^2-p)(p^2-1) & : \text{elem} \end{cases}$ HW

Example
Groups of order 45 + a condition.

Let $|G| = 45 = 3^2 \cdot 5$ & suppose $P \trianglelefteq G$ w/ $|P| = 9 = 3^2$

Claim
G is abelian.

Pf/ $G \circlearrowright P$ by conjugation
$g * p = gpg^{-1} \in P \leftarrow$ normal
$\ker = \{g \mid gp = pg \ \forall p \in P\}$
$= C_G(P)$

$G \longrightarrow \mathrm{Aut}(P)$ P.R.
$G \to G/C_G(P) \to \mathrm{Aut}(P)$ s.i.t.

LaGrange
$|G/C_G(P)| \,\big|\, |\mathrm{Aut}(P)|$
-1 or 5 $\longrightarrow$ 6 or 48

$|P| = 9 = 3^2 \Longrightarrow$ $\mathrm{Aut}(P) = \begin{cases} (\mathbb{Z}/9\mathbb{Z})^\times & : \text{order } 6 \\ GL_2(\mathbb{F}_3) & : 48 \end{cases}$

$|G| / |C_G(P)|$
$|G|$: 45; $|C_G(P)|$: Find, 9 or 45

$|P| = 3^2 \Rightarrow P$ abelian
$P \leq C_G(P)$
LG $9 \,\big|\, |C_G(P)| \,\big|\, 45$
$|C_G(P)|$ must be 9 or 45

$\Longrightarrow G = C_G(P)$

$\Longrightarrow P \leq Z(G) \leq G$

$\Longrightarrow |Z(G)| = 45$ or 9

$|Z(G)| = 45$: $Z(G) = G \Rightarrow G$ abelian ✓

$|Z(G)| = 9$: $|G/Z(G)| = \frac{45}{9} = 5$
$\Rightarrow G/Z(G) \simeq \mathbb{Z}_5$ cyclic
$\overset{HW}{\Longrightarrow} G$ abelian ✓

① Know about groups of order $9 = 3^2$
② P was maximally 3-power ordered.
i.e. $\gcd(9, 5) = 1$
③ P was normal.

Defn
$p^\alpha | n$ is a maximal p-divisor if
* $p^{\alpha+1} \nmid n$
* $n = p^\alpha \cdot m$ w/ $(p^\alpha, m) = 1$

$n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. $\quad p_1 \neq p_2 \neq \cdots \neq p_n$
The max'l p-divs are
* $p_i^{\alpha_i}$

$|G| = n$ Looking for $H \leq G$ w/ $|H| =$ max'l p-divisor of n.

Sylow $\Rightarrow$ These exist
$\Rightarrow$ How many?
$\Rightarrow$ They are conj.