Recall
$G \curvearrowright G$ $g * a = gag^{-1}$

$*\ |G * a| = |G : C_G(a)|$

Study conjugation on S_n

Recall
Prop: $\sigma \in S_n$, $\tau \in S_n$
$\tau * \sigma = \tau\sigma\tau^{-1}$

$\tau * \sigma(\tau(i)) = \tau(j)$
$\iff \sigma(i) = j$

Pf/ $\tau\sigma\tau^{-1}(\tau(i)) = \tau(\sigma(i))$
$= \tau(j)$

Corollary $\sigma, \tau \in S_n$
$\sigma = (i_1\ i_2 \cdots)(j_1\ j_2 \cdots) \cdots$
Then
$\tau\sigma\tau^{-1} = (\tau(i_1)\ \tau(i_2) \cdots)(\tau(j_1)\ \tau(j_2) \cdots)$

Ex $\sigma = (1\ 2)(3\ 4\ 5) \in S_5$
$\tau = (1\ 2\ 3\ 4)$

$\tau\sigma\tau^{-1} = (\tau(1)\ \tau(2))$
$(\tau(3)\ \tau(4)\ \tau(5))$
$\tau * \sigma = (2\ 3)(4\ 1\ 5)$
$\sigma = (1\ 2)(3\ 4\ 5)$

Ex do for
$\sigma = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$
$\tau = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$

Note
$\sigma = (\cdots n_1\ \text{terms})(\cdots n_2) \cdots$
$\tau * \sigma = (n_1\ \text{terms})(\cdots n_2) \cdots$

Def $\sigma \in S_n$ decomposes into $\sigma_1\ \sigma_2\ \sigma_3 \cdots \sigma_m$ disjoint n_i-cycles (including 1-cycles) & $n_1 \le n_2 \le \cdots \le n_m$.
Then cycle type of σ is $(n_1, n_2, n_3, \cdots, n_m)$

Ex $(1\ 2)(3\ 4\ 5)$
type $(2, 3)$ ← $2+3=5$
$(5)(1\ 2\ 3\ 4)$
type $(1, 4)$ ← $1+4=5$

Theorem
2 elements of S_n are conjugate
$\iff$ same cycle type

Pf/ $\Rightarrow$ ✓
$\Leftarrow$ pf by example

$\sigma_1 = (1\ 2)(3\ 4\ 5)$
$\sigma_2 = (2\ 5)(1\ 4\ 3)$

1 2 3 4 5
2 5 1 4 3
τ

$\tau = (1\ 3\ 5\ 2)$
$\sigma_1 = \tau\sigma_2\tau^{-1}$ ✓
$\varnothing$

Ex/ τ not unique
$\tau_1 = (2\ 3) \rightsquigarrow (1\ 4)\sigma_2$
$\tau_1 = (2\ 1)(3\ 4)$
$\tau_2 = (2\ 1\ 3\ 4)$
ck $\tau_1\sigma_1\tau_1^{-1} = \sigma_2 = \tau_2\sigma_1\tau_2^{-1}$

Def^n $n \in \mathbb{Z}_{>0}$.
A partition of n is $\{n_1, n_2, \cdots, n_m\} \in \mathbb{Z}_{>0}$
w/ $n_1 + n_2 + \cdots + n_m = n$

Prop $\sigma \in S_n$ type $(n_1, \cdots, n_m)$ ← a partition of n.
Pf/ Each # $1, \dots, n$ appears in exactly one cycle $\varnothing$

Thm $\exists$ bijection
$\left\{\begin{matrix}\text{conjug.}\\ \text{classes}\\ \text{in } S_n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{partitions}\\ \text{of}\\ n\end{matrix}\right\}$
$S_n * \sigma \longmapsto$ cycle type of σ
$(1\ 2 \cdots n_1)$ $(n_1{+}1, \cdots, n_1{+}n_2)$ $\cdots$ $\longleftarrow (n_1, n_2, \cdots)$

Conjugacy classes of S_5

Partitions	Conj. Rep
1+1+1+1+1	$(1)(2)(3)(4)(5) = 1$
1+1+1+2	$(1\ 2)$
1+1+3	$(1\ 2\ 3)$
1+4	$(1\ 2\ 3\ 4)$
5	$(1\ 2\ 3\ 4\ 5)$
1+2+2	$(1\ 2)(3\ 4)$
2+3	$(1\ 2)(3\ 4\ 5)$

Computing Centralizers
Let $\sigma = m$-cycle
$|S_n * \sigma| = \#\{m\text{-cycles}\}$
$= \dfrac{n \cdot (n-1)(n-2) \cdots (n-m+1)}{m}$
$= \dfrac{1}{m}\dfrac{n!}{m!}$
But $|S_n * \sigma| = \dfrac{|S_n|}{|C_G(\sigma)|}$ ← $n!$
Solve for
$|C_{S_n}(\sigma)| = m(n-m)!$

What is $C_{S_n}(\sigma)$.
= elts τ commute w/ σ.
What elts comute w/ σ
1) $\underbrace{1, \sigma, \sigma^2, \dots, \sigma^{m-1}}_{m}$
2) τ disjoint. i.e. τ fix elts in σ.
$\tau \in S_{n-m}$ $|S_{n-m}| = (n-m)!$

Prop
$C_{S_n}(\sigma) = \{\sigma^i \tau \mid \tau \in S_{n-m}\}$
m $(n-m)!$

Ex $\sigma = (1\ 3\ 5) \in S_5$
$C_{S_5}(\sigma) = \left\{ \begin{matrix}(1\ 3\ 5)^i \cdot \tau \\ \tau \in S_{\{2,4,6,7\}}\end{matrix} \right\}$

Lemma $H \trianglelefteq G$.
$K \subseteq G$ conjug. class
$\Rightarrow K \cap H = \varnothing$ or $K \subseteq H$
Pf/ $x \in K \cap H$
$gxg^{-1} \in H\ \ \forall g \in G$
$\uparrow$ K all of these. $\varnothing$

Slogan
Normal subs are unions of conj. classes

Theorem
A_5 is simple.

Remark:
$\sigma_2 = \tau * \sigma_1$ in S_5
$\not\Rightarrow$ $\sigma_2 = \tau * \sigma_1$ w/ $\tau \in A_5$
i.e./ τ may be odd

Pf/ cycle types

1,1,1,1,1	(1) ← $\lvert K_0\rvert = 1$
1,1,3	$(1\ 2\ 3)$ ← $\lvert K_1\rvert = 20$
5	$(1\ 2\ 3\ 4\ 5)$ ← $\lvert K_2\rvert = 12$, $\lvert K_3\rvert = 12$
1,2,2	$(1\ 2)(3\ 4)$ ← $\lvert K_4\rvert = 15$

classes of type 1,1,3
$C_{S_5}(1\ 2\ 3) = \{(1\ 2\ 3)^i \cdot \tau$
$\tau \in S_{\{4,5\}}\}$
$\uparrow$ transp so odd

$\Rightarrow C_{A_5}(1\ 2\ 3) = \langle(1\ 2\ 3)\rangle$
$|C_{A_5}(1\ 2\ 3)| = 3$
$|A_5 * (1\ 2\ 3)| = \dfrac{|A_5|}{3} = 20$
But 20 3-cycles
$\dfrac{5 \cdot 4 \cdot 3}{3} = 20$

classes of 5-cycles
$C_{S_5}((1\ 2\ 3\ 4\ 5)) = \langle(1\ 2\ 3\ 4\ 5)\rangle$
$\Rightarrow C_{A_5}(1\ 2\ 3\ 4\ 5) = \langle(1\ 2\ 3\ 4\ 5)\rangle$
$|C_{A_5}(1\ 2\ 3\ 4\ 5)| = 5$
$|A_5 * (1\ 2\ 3\ 4\ 5)| = \dfrac{60}{5} = 12$
How many 5 cycles?
$\dfrac{5!}{5} = 24$
check
$\tau(1\ 2\ 3\ 4\ 5)\tau^{-1}$
$= (1\ 3\ 5\ 2\ 4)$
τ odd

So far
K_0 K_1 K_2 K_3
$1 + 20 + 12 + 12 = 45$
15 left
all $(a\ b)(c\ d) = \sigma$
$\sigma^2 = 1$
1) $(1\ 2)(3\ 4)$ commutes w/ $(1\ 3)(2\ 4)$ & $(1\ 4)(2\ 3)$
But nothing else. (1)
$|C_{A_5}((1\ 2)(3\ 4))| = 4$
$|A_5 * ((1\ 2)(3\ 4))| = \dfrac{60}{4} = 15$

K_0	K_1	K_2	K_3	K_4
1	20	12	12	15

Suppose $H \trianglelefteq G$
1) $H = \bigcup K_i \Longrightarrow |H| = \Sigma$
$|H| \mid 60$ ← Lagranges
$|H| \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
$|H| = 1 + \delta_1 \cdot 20 + \delta_2 \cdot 12 + \delta_3 \cdot 12 + \delta_4 \cdot 15$
$\delta_i = 0$ or 1
$\forall\ \delta_i = 0$ $H = 1$ $\varnothing$
or $\forall\ \delta_i = 1$ $H = A_5$