Recall

$G \times G \to G$ Left mult
$g \cdot a \mapsto ga$

Actions by conjugation

$G \curvearrowright G$ $g * a = gag^{-1}$

Lemma
This is an action.

Pf/ $1 * a = 1a1^{-1} = a$

$g_1 * (g_2 * a) = g_1 * (g_2 a g_2^{-1})$
$= g_1 (g_2 a g_2^{-1}) g_1^{-1}$
$= (g_1 g_2) a (g_2^{-1} g_1^{-1})$
$= (g_1 g_2) a (g_1 g_2)^{-1}$
$= (g_1 g_2) * a$

Defn
* $a, b \in G$ are conjugate if $\exists g \in G$ w/ $b = gag^{-1}$
* Orbits called conjugacy classes

Ex/
* G abelian
$\Rightarrow g * a = gag^{-1} = agg^{-1} = a$
Action is trivial
$G * a = \{a\}$
* G never acts transitively by conjug. if $|G| > 1$.
Pf/ $G * 1 = \{1\}$ $\quad g \cdot 1 \cdot g^{-1} = g \cdot g^{-1} = 1$
* $S_3 \curvearrowright S_3$ by conj
Classes
$\{\{1\}, \{(12), (13), (23)\}, \{(123), (132)\}\}$
1 $\quad$ 3 $\quad$ 2

Def $S \subseteq G$ subset
$gSg^{-1} = \{gxg^{-1} \mid x \in S\}$
Defn $P(G) = \{S \subseteq G\}$ power set
Get a group action
$G \curvearrowright P(G)$,
$g * S = gSg^{-1}$
Computations
$|G * S| = \#\{S' \text{ s.t. } S' = gSg^{-1}\}$
$\quad \parallel$
$|G : G_S| = |G : N_G(S)|$
$G_S = \{g \in G \mid gSg^{-1} = S\}$
$= N_G(S)$

Prop
$\#\{\text{sets conj. to } S \subseteq G\}$
$= |G : N_G(S)|$
$\#\{\text{conjugates of } a \in G\}$
$= |G : C_G(a)|$
$C = N_G(\{a\})$

Pf/ $C_G(a) = \{g \in G \mid ga = ag\}$
$= \{g \in G \mid gag^{-1} = a\}$
$= N_G(\{a\})$

So
$|G * a| = |G| / |C_G(a)|$
this $\quad$ this sub.

Remark
$a \in Z(G) \to ag = ga \ \forall g$
$gag^{-1} = a \quad C_G(a) = G$
$G * a = \{a\}$

Recall $G \curvearrowright A$ fin.
Orbits $O_1, \dots, O_r$
$|A| = |O_1| + |O_2| + \dots + |O_r|$

Thm (Class Equation)
G finite group.
$g_1, \dots, g_r \in G$ in the distinct conjugacy classes of G which are not in $Z(G)$
$\Longrightarrow$
$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$
Pf/ Conjugacy classes are
$\{z_1\}, \dots, \{z_n\}$ for $z_i \in Z(G)$
& $G * g_i \quad i = 1, \dots, r$
But $|\{z_i\}| = 1$
$|G * g_i| = |G : C_G(g_i)|$
Adding them up we win

Ex/ G abel.
$|G| = |Z(G)|$
* $D_8$ $\quad Z(D_8) = \{1, r^2\}$
$\rightsquigarrow D_8 * r$.
$|D_8 : \langle r \rangle| = 2$
$\langle r \rangle \leq C_G \langle r \rangle \lneq D_8$
$4 \quad = 4 \quad\quad 8$
$|D_8 : C_G(r)| = 2$
$|D_8 * r| = 2$
Notice $r$ in there
$srs^{-1} = r^{-1} \cancel{s\,s^{-1}}$
$= r^3$
$D_8 * r = \{r, r^3\}$
others $\{s, sr^2\}$
$\{sr, sr^3\}$

$|D_8| = |Z(D_8)| + \sum |G * g|$
$\parallel$
$8 \quad = 2 + 2 + 2 + 2$
$= 8$

Class equation in action

Thm $|G| = p^\alpha$ p prime.
$\Rightarrow Z(G) \neq 1$.

Pf/ $|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$
div by p $\quad$ div by p $\quad$ this is div by p ✓

$|G : C_G(g_i)| \cdot |C_G(g_i)| = |G| = p^i$
$g_i \neq Z(G)$
$\Longrightarrow |G : C_G(g_i)| > 1$
so $p \mid |G : C_G(g_i)| = p^\alpha$
$p \mid |Z(G)| \geq 0$
$1 \in Z(G)$
$\Longrightarrow |Z(G)| \geq p$

Thm (gps order $p^2$)
$|G| = p^2$, then G abelian
& $G \simeq \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$
Pf/ $|Z(G)| \neq 1$
$\Longrightarrow |Z(G)| = p$ or $p^2$ abelian
$|G/Z(G)| = \frac{|G|}{|Z(G)|} = \frac{p^2}{p} = p$
$G/Z(G) \simeq \mathbb{Z}_p$
HW $\Rightarrow$ G abelian

If $\exists x \in G$ w/ $|x| = p^2$
$\Longrightarrow G = \langle x \rangle = \mathbb{Z}_{p^2}$ done
Else $\forall x \neq 1$, $|x| = p$.
Take $y \in G \setminus \langle x \rangle$, $|y| = p$
$\langle x \rangle \times \langle y \rangle \xrightarrow{\sim} \mathbb{Z}_p \times \mathbb{Z}_p$
$(x^i, y^j)$
$\downarrow$
$G \quad x^i y^j$
check
surj ✓
inj ✓ ex.

Intro Conjugation on $S_n$.

Thm: $\sigma \in S_n$
$\tau * \sigma = \tau \sigma \tau^{-1}$
$\sigma(i) = j$
$\tau * \sigma(\tau(i)) = \tau(j)$
Pf/ $\tau * \sigma(\tau(i)) =$
$\tau \sigma \tau^{-1}(\tau(i))$
$= \tau \sigma(i)$
$= \tau(j)$ ✓

Observe
$\sigma = (1\,3)(4\,5\,2)$
$\quad 2 \quad\quad 3$
$\tau \sigma \tau^{-1} = (\tau(1)\ \tau(3))(\tau(4)\ \tau(5)\ \tau(2))$
$\quad 2 \quad\quad\quad 3$

MEAT

Conjugates are those w/ same shape of decomp