$\varepsilon: S_n \longrightarrow \{\pm 1\}$ "sign"
1) ε is a hom
2) $\varepsilon(i\ j) = -1$

Def $\sigma \in S_n$
σ is even if $\varepsilon(\sigma) = 1$
σ is odd if $\varepsilon(\sigma) = -1$

$\sigma = (a_1\ a_2 \cdots a_m) \in S_n$
$= (a_1 a_m)(a_1 a_{m-1}) \cdots (a_1 a_2)$
$\varepsilon(\sigma) = \varepsilon(a_1 a_m) \cdots \varepsilon(a_1 a_2)$
$= (-1)(-1)\cdots(-1)$
$= (-1)^{m-1}$

Ex/ $\sigma = (1\,2\,4)(3\,5) \in S_5$
$\varepsilon(\sigma) = \varepsilon(1\,2\,4)\,\varepsilon(3\,5)$
$= (-1)^2 \cdot (-1) = -1$

Def (Alternating Gp)
$A_n = \ker \varepsilon$
$= \{\text{even perms}\} \le S_n$

Thm
1) $S_n / A_n \cong \{\pm 1\} \simeq \mathbb{Z}_2$
2) $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Example
A_3
$\varepsilon(1\,2\,3) = 1$
$(1\,2\,3) \in A_3$
so is $(1\,2\,3)^2 = (1\,3\,2)$
$\varepsilon(1\,2) = -1$ $(1\,2) \notin A_3$
$A_3 = \{(1), (1\,2\,3), (1\,3\,2)\}$

Thm
A_n is simple if $n \ge 5$

Thm (Cauchy) $|G| = n$
$p \mid n$ p prime $\Rightarrow$
$\exists\, H \le G$ $|H| = p$

Q. Can we eliminate prime?

Ex/ $|A_5| = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2} = 60$
$30 \mid 60$
if $H \le A_5$ $|H| = 30$
$|A_5 : H| = 2 \Rightarrow H \trianglelefteq A_5$ ↯

Group Actions

Def $G \circlearrowright A$ G gp, A set

$G \times A \longrightarrow A$, $(g,a) \longmapsto g\cdot a$
1) $1 \cdot a = a$
2) $g\cdot(h\cdot a) = (gh)\cdot a$

$g \in G$. Get
$\sigma_g : A \longrightarrow A$
$\sigma_g(a) = g \cdot a$

$G \longrightarrow S_A = \mathrm{Aut}(A)$
$g \longmapsto \sigma_g$ a hom

Prop
$\{\text{Actions } G \circlearrowright A\} \longleftrightarrow \{\text{homs } G \to S_A\}$
$g\cdot a = \phi(g)(a) \longleftarrow \phi$

Def * Kernel of action
$= \{g \in G \mid g\cdot a = a\ \forall a\}$
$= \ker \phi$ ← HW

* $a \in A$. stabilizer of a is
$G_a := \{g \in G \mid g\cdot a = a\}$

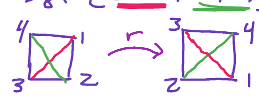
* An action is faithful if $\ker\phi = 1$

Ex/ * $S_n \circlearrowright \{1, \cdots, n\}$
$\sigma \cdot i = \sigma(i)$
HW $(S_n)_i \simeq S_{n-1}$

* $D_8 \circlearrowright \{1,2,3,4\}$

$rs \cdot 1 = 1$
$rs \cdot 2 = 4$ . . .

* $D_8 \circlearrowright \{\{1,3\},\{2,4\}\}$



$r(\backslash) = /$
$r(/) = \backslash$
$rs(\backslash) = \backslash$
$rs(/) = /$

$rs \in \mathrm{Ker}$
Action not faithful

Thm $G \circlearrowright A$
1) $a \sim b$ if $b = g\cdot a$ some g is an equiv. rel.
2) Equiv classes are the orbits $G\cdot x = \{g\cdot x \mid g \in G\}$
3) $|G\cdot x| = |G : G_x|$
$|G| < \infty \Rightarrow = |G| / |G_x|$
set, Groups

Pf/ $G\cdot x \xrightarrow{\Phi} \{\text{cosets of } G_x\}$
$b = g\cdot x \longmapsto gG_x$

Well def
$b = g\cdot x = h\cdot x \rightsquigarrow gG_x = hG_x$
$h^{-1}g\cdot x = h^{-1}h\cdot x = x \Longrightarrow h^{-1}g \in G_x$

Inj
If $\phi(b) = \phi(c)$
$\Longrightarrow b = c$
gx hx

Surj
gG_x some coset.
$g\cdot x \in G\cdot x$
$\phi(g\cdot x) = gG_x$ ☒

Thm $\sigma \in S_n$
Then σ factors uniquely into disjoint cycles
← up to reordering
$(1\,3)(2\,4) = (2\,4)(3\,1)$

Pf/ $S_n \circlearrowright \{1, \cdots, n\} = A$
$\sigma \in S_n$
$\langle \sigma \rangle = G \le S_n$
$G \circlearrowright A$
Get partition of A into disjoint orbits

$\mathcal{O} = \{x, \sigma x, \sigma^2 x, \ldots\}$
$\mathcal{O} \longleftrightarrow \{\text{cosets of } G_x\} \simeq G/G_x \simeq \mathbb{Z}_d$
also cyclic order d
$\sigma^i x \longmapsto \sigma^i G_x$
G is cyclic, so $G_x \trianglelefteq G$

$\mathcal{O} \longleftrightarrow G/G_x$
$\{x, \sigma x, \ldots, \sigma^{d-1}x\} = \{G_x, \sigma G_x, \ldots, \sigma^{d-1} G_x\}$

$\sigma^d x = x$

σ acts like
$(x\ \sigma x\ \sigma^2 x \ldots \sigma^{d-1} x)$
Do each (disjoint) orbit
Get disjoint cycles
$\Longrightarrow \sigma$ decomposes as needed.

Unique:
Say σ has cycle $(x_1\ x_2 \cdots x_m)$
$\sigma(x_1) = x_2$
$\sigma(x_2) = x_3$
⋮
$\sigma(x_m) = x_1$

$G\cdot x_1 = \{x_1, \ldots, x_m\}$ ☒