Def^n

$|G| = p^\alpha \cdot m$ where $p \nmid m$

RMK: P-grp, p-prime

- A subgroup $P \le G$ w/ $|P| = p^\alpha$ is Sylow p-subgroup.
- The set of them is $Syl_p(G)$
- The number of Sylow p-subs is $n_p = |Syl_p(G)|$.

Theorem

$|G| = p^\alpha m$ w/ $p \nmid m$.

1) Sylow p subs exist.
2) $P, Q \in Syl_p(G)$, then $Q = gPg^{-1}$ some $g \in G$. [so $P \simeq Q$]
3) $n_p = 1 + kp \equiv 1 \mod p$.

& if $P \in Syl_p(G) \Rightarrow n_p = |G : N_G(P)|$.

So $n_p \mid m$

Pf/ $n_p = |G : N_G(P)|$ $= |G| / |N_G(P)|$ $= \dfrac{p^\alpha m}{p^\alpha t}$ $= m/t$ $\Rightarrow n_p \cdot t = m \Rightarrow n_p \mid m$

$P \le N_G(P)$ so L.G. $p^\alpha \mid |N_G(P)|$

Example $|G| = 45 = 3^2 \cdot 5$

$Syl_3(G)$ ie. subgroups order 9.

$n_3 = 1 + k \cdot 3 \in \{1, \cancel{4}, \cancel{7}, \cancel{10}, \cancel{13}, \dots\}$

$n_3 \mid 5 \Rightarrow n_3 = 1$

Let $P \le G$ w/ $|P| = 9$

↑ P only subgroup of order 9.

But $|gPg^{-1}| = 9 \Leftarrow$

$\Rightarrow gPg^{-1} = P \Rightarrow P \trianglelefteq G$.

Monday $\Rightarrow G$ abelian.

Next $\Rightarrow G \simeq \mathbb{Z}_{45}$ or $\mathbb{Z}_3 \times \mathbb{Z}_{15}$

Prop $P \le G$ is a Sylow-p subgroup. Then

$P \trianglelefteq G \iff n_p = 1$

Pf/ ($\Rightarrow$) If $Q \in Syl_p(G)$

$\Rightarrow Q = gPg^{-1} = P$

so $Syl_p(G) = \{P\} \Rightarrow n_p = 1$.

($\Leftarrow$) $n_p = 1$.

Know $gPg^{-1} \in Syl_p(G)$

$\Rightarrow \; = P$

$\Rightarrow P \trianglelefteq G$ $\varnothing$

Examples

* G abelian. Then there is a unique Sylow p-subgroup for every p.

Pf/ $P \in Syl_p(G)$

$\Rightarrow P \trianglelefteq G \Rightarrow n_p = 1$

* $|S_3| = 2 \cdot 3 = 6$

$Syl_2(S_3) = \{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\}$

- All conjugate!
- All iso $\simeq \mathbb{Z}_2$
- $n_2 = 3 \equiv 1 \mod 2$ & $3 \mid 3$

$Syl_3(S_3) = \{\langle(123)\rangle\}$

$n_3 = 1$ & $\langle(123)\rangle \trianglelefteq S_3$

* $|D_8| = 8 = 2^3$

$Syl_2(D_8) = \{D_8\}$.

* $|S_4| = 4! = 2^3 \cdot 3$

$Syl_3(S_4) \leftarrow$ order 3

$n_3 = \{\cancel{1}, 4, \cancel{7}, \cancel{10}, \dots\}$

$n_3 \mid 8$

$\{\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle\}$

$Syl_2(S_4) \leftarrow$ order $2^3 = 8$

Claim! All Sylow 2-subs of S_4 are $\simeq D_8$.

Pf/ $D_8 \xrightarrow{\phi} S_4$

$\sigma \longmapsto \sigma$ on vertices, $\sigma \in S_4$.

$D_8 \simeq \phi(D_8) \le S_4$

↑ order 8, so a Sylow 2-sub.

$D_8 \le S_4$ is it normal?

$\iff$ What is $n_2 = \{1, 3, \cancel{5}, \cancel{7}, \dots\}$

$n_2 \mid 3$

Claim $n_2 = 3$

Pf/ $\exists\ \tau \in S_4$, $r = (1234)$

$\tau r \tau^{-1} = (1342) \leftarrow$

↑ This not in D_8 $\varnothing$

$\Rightarrow D_8 \ntrianglelefteq S_4$

$\Rightarrow n_2 \ne 1$ $\varnothing$

$|G| = p \Rightarrow G \simeq \mathbb{Z}_p$

$|G| = p^2 \Rightarrow G \simeq \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$

Applications: Groups order pq

$|G| = pq$ w/ $p < q$ · prime

Fix $P \in Syl_p(G)$, $Q \in Syl_q(G)$. } Exist by Sylow.

So $|P| = p$, $|Q| = q$.

Claim: $Q \trianglelefteq G$.

Pf/ $n_q = 1 + kq$

$n_q \mid p$ so 1 or p

$n_q \ne 1 \Rightarrow n_q > q > p$ so $n_q \nmid pq$

$\Rightarrow n_q = 1 \Rightarrow Q \trianglelefteq G$ $\varnothing$

Claim

$P \trianglelefteq G \iff G \simeq \mathbb{Z}_{pq}$.

Pf/ ($\Leftarrow$) trivial.

($\Rightarrow$) $P \trianglelefteq G$ then $G \circlearrowright P$ by conjugation $g * x = gxg^{-1}$. | $Ker = C_G(P)$

Get $G \longrightarrow Aut(P) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ ↑ order $p-1$

$\downarrow$ $\nearrow$

$G/C_G(P)$

↑ order p, q or 1

$p, q, pq > p-1$ so L.G.

$|G/C_G(P)| = 1$

$\Rightarrow G = C_G(P)$

ie. $P \le Z(G)$.

$P = \langle x \rangle$, $Q = \langle y \rangle$.

So $xy = yx$

Lemma If $xy = yx$ & $\gcd(|x|, |y|) = 1$

$\Rightarrow |xy| = |x||y|$.

Pf/ Exercise.

$\Rightarrow |xy| = |x||y| = pq$

So G has an elt of order pq

$G = \langle xy \rangle \Rightarrow G = \mathbb{Z}_{pq}$ $\varnothing$

Corollary $p \nmid q-1$

$\Rightarrow G \simeq \mathbb{Z}_{pq}$

Pf/ $p \nmid q-1 \Rightarrow n_p = 1 + kp \nmid q$ (else $kp \mid q-1$)

unless $k = 0$.

So $n_p = 1 \Rightarrow P \trianglelefteq G$