$|G| = pq$ $p<q$

$p \mid q-1$

Last time

$P \in Syl_p$, $Q \in Syl_q$

$P \cong \mathbb{Z}_p$, $Q \cong \mathbb{Z}_q$

S. $Q \trianglelefteq G$,

Lagrange $P \cap Q = 1$.

Know $PQ = G$

So $G \cong Q \rtimes P$

Classified by maps

$P \longrightarrow Aut(Q)$

$P \cong \mathbb{Z}_p$, $Aut(Q) \cong \mathbb{Z}_{q-1}$

use $p \mid q-1$ unique order p

$H = \langle \gamma \rangle$, $\gamma^p = 1$

Reduced to maps

$\mathbb{Z}_p \xrightarrow{\phi_i} \langle \gamma \rangle = H$

$\mathbb{Z}_p = \langle x \rangle$, $x \longmapsto \gamma^i$

There are p of these

$G_i = Q \rtimes_{\phi_i} P$

$i = 0$, ϕ_i trivial

$\Longrightarrow G_0 \cong Q \times P = \mathbb{Z}_{pq}$

Claim: $i = 1, \dots, p-1$

Then $G_i \cong G_j$

Pf/ $P \xrightarrow{\phi_i, \phi_j} Aut\, Q$.

$x \longmapsto \gamma^i, \gamma^j$

Remark: $\gamma^j = (\gamma^i)^k$

Define $\psi : P \longrightarrow P$

$x \longmapsto x^k$

Get a diagram

$$\begin{tikzcd} P \arrow[d, "\psi"] \arrow[dr, "\phi_j"] & \\ P \arrow[r, "\phi_i"] & Aut\, Q \end{tikzcd}$$

$\phi_i \circ \psi = \phi_j$

$\phi_i \cdot \psi(x) = \phi_i(x^k)$

$= \phi_i(x)^k$

$= (\gamma^i)^k$

$= \gamma^j$

$= \phi_j(x)$.

$G_j \xrightarrow{\Phi} G_i$

$Q \rtimes_{\phi_j} P \xrightarrow{\sim} Q \rtimes_{\phi_i} P$

$(q, x^a) \longmapsto (q, x^{k \cdot a})$

$\Phi((q_1, x^a)(q_2, x^b))$

$= \Phi(q_1 \phi_j(q_2)^a, x^a x^b)$

$= \Phi(q_1 \gamma^j(q_2)^a, x^a x^b)$

$= (q_1 \gamma^j(q_2)^a, x^{ka} x^{kb})$

$\Phi(q_1, x^a)\Phi(q_2, x^b)$

$= (q_1, x^{ka})(q_2, x^{kb})$

$= (q_1 \phi_i(q_2)^{ka}, x^{ka} x^{kb})$

$= (q_1 \gamma_i^k(q_2)^a, x^{ka} x^{kb})$

$= (q_1 \gamma_j(q_2)^a, x^{ka} x^{kb})$

Prop: $|G| = pq$

Then

$p \nmid q-1$ $G \cong \mathbb{Z}_{pq}$

Else

$p \mid q-1 \Rightarrow G \cong \mathbb{Z}_{pq}$

or

$G \cong \mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$

for $\phi : \mathbb{Z}_p \longrightarrow Aut(\mathbb{Z}_q)$

nontrivial.

i.e. $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ is the unique nonab gp order $\mathbb{Z}_{pq}$.

Consequences

Groups order 6

$\boxed{\mathbb{Z}_6 \text{ \& } S_3}$

Groups of order 10

$\boxed{\mathbb{Z}_{10} \text{ \& } D_{10}}$

Groups order $2q$ - q prime

$\boxed{\mathbb{Z}_{2q} \quad D_{2q}}$

This begs a natural question:

$\left\{ \begin{matrix} \text{Maps} \\ K \xrightarrow{\phi} Aut\, H \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Semidirect} \\ \text{product} \\ \text{group} \end{matrix} \; H \rtimes_\phi K \right\}$

up to iso.

Previous argument shows not bijective.

Rmk A, B

$\{\text{Maps } A \to B\}$

$= Hom(A, B) \longleftarrow$ a gp

$\phi, \psi \rightsquigarrow \phi * \psi(a)$

$= \phi(a)\psi(a) \in B$.

$\left\{ \begin{matrix} \text{Maps} \\ \mathbb{Z}_p \to Aut(\mathbb{Z}_q) \end{matrix} \right\} \longleftrightarrow \{\mathbb{Z}_q \rtimes \mathbb{Z}_p\}$

p elts 2.

Question

Suppose $\phi, \psi : K \longrightarrow Aut(H)$

When is

$H \rtimes_\phi K \cong H \rtimes_\psi K$.

Lemma $\phi, \psi : K \longrightarrow Aut(H)$

Suppos $\exists\ \gamma \in Aut\, K$

s.t.

$$\begin{tikzcd} K \arrow[d, "\gamma"] \arrow[dr, "\phi"] & \\ K \arrow[r, "\psi"] & Aut\, H \end{tikzcd}$$

Then

$H \rtimes_\phi K \xrightarrow{\sim} H \rtimes_\psi K$

$(h, k) \longmapsto (h, \gamma(k))$.

Example

$|G| = 12$. $P \in Syl_2$, $Q \in Syl_3$

Assume

$Q \trianglelefteq G$.

$P \cap Q = 1$ (Lagrange)

$PQ = G$

$\Longrightarrow G \cong Q \rtimes P$

Notice $|P| = 4$

$|Q| = 3 \longleftarrow Q \cong \mathbb{Z}_3$

$P \cong \mathbb{Z}_4$ ①

or

$P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ ②

① $P \cong \mathbb{Z}_4$ $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

Maps $\mathbb{Z}_4 \longrightarrow Aut(\mathbb{Z}_3)$

$\mathbb{Z}_4 = \langle x \rangle$, $Aut(\mathbb{Z}_3) \cong \mathbb{Z}_2 = \{id, \iota\}$

$x \longmapsto id \longleftarrow \mathbb{Z}_3 \times \mathbb{Z}_4$

$x \longmapsto \iota \longleftarrow \mathbb{Z}_3 \rtimes \mathbb{Z}_4$

② $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$

$\mathbb{Z}_3 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Classify

$\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow Aut\, \mathbb{Z}_3 = \{id, \iota\}$

$a, b \longmapsto id$ or ι

Both go to id

$\Longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

3 maps left

ϕ_1	ϕ_2	ϕ_3
$a \mapsto \iota$	$a \mapsto id$	$a \mapsto \iota$
$b \mapsto \iota$	$b \mapsto \iota$	$b \mapsto id$

swap a & ab swap a & b

$\forall\ i, j$, $\exists\ \gamma \in Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)$

s.t.

$$\begin{tikzcd} \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[d, "\gamma"] \arrow[dr, "\phi_i"] & \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r, "\phi_j"] & Aut(\mathbb{Z}_3) \end{tikzcd}$$

So

$\mathbb{Z}_3 \rtimes_{\phi_i} (\mathbb{Z}_2 \times \mathbb{Z}_2)$

all agree

This must be D_{12}

Pf/ $D_{12} \not\cong A_4$

$\Longrightarrow Q \trianglelefteq D_{12}$

$D_{12} \not\cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ last time

This is all that's left.

Groups order 30

Reminder

$\mathbb{Z}_2 \longrightarrow \mathbb{Z}_n$

$x \longmapsto \iota$

Induces $\mathbb{Z}_n \rtimes \mathbb{Z}_2 \cong D_{2n}$

Know $H \trianglelefteq G$ $H \cong \mathbb{Z}_{15}$

Know $\exists\ K \leq G$ $|K| = 2$

$K \cong \mathbb{Z}_2$

Lagrange $H \cap K = 1$

$\Longrightarrow G \cong H \rtimes K$

$= \mathbb{Z}_{15} \rtimes \mathbb{Z}_2$

Reduces Finding Maps

$\mathbb{Z}_2 \longrightarrow Aut(\mathbb{Z}_{15})$

Amounts to picking an element of order 1 or 2 in $Aut\, \mathbb{Z}_{15}$

Lemma (HW)

H, K Groups.

$|H| = m$. $|K| = n$. $\gcd(m, n) = 1$.

$\Longrightarrow Aut(H \times K) \cong Aut(H) \times Aut(K)$

Consequence

$Aut(\mathbb{Z}_{15}) \cong Aut(\mathbb{Z}_3 \times \mathbb{Z}_5)$

$\cong Aut(\mathbb{Z}_3) \times Aut(\mathbb{Z}_5)$

$\cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$

Gps order 30

$\Updownarrow$

Maps $\mathbb{Z}_2 \longrightarrow Aut(\mathbb{Z}_{15})$

$\Updownarrow$

Elts order 1 or 2 in $Aut(\mathbb{Z}_{15})$

$\Updownarrow$

Elts order 2 in $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$

$\Updownarrow$

$\{(0,0), (1,0), (0,2), (1,2)\}$.

$(\leq 4$ gps order $30)$

Table of Stuff

$\lvert G\rvert = p$	$\lvert G\rvert = p^2$
$\boxed{G \simeq \mathbb{Z}_p}$	$\boxed{G \simeq \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_p \times \mathbb{Z}_p}$

Yellow Box: = Complete Classification

$\lvert G\rvert = pq$

$P \in Syl_p$, $Q \in Syl_q$

* G abelian $\Rightarrow$ $\boxed{G \simeq \mathbb{Z}_{pq}}$
* $Q \trianglelefteq G$
* If $P \trianglelefteq G$
* If $p \nmid q-1$
* If $p \mid q-1$

$\boxed{G \simeq \mathbb{Z}_{pq} \text{ or } \mathbb{Z}_q \rtimes \mathbb{Z}_p}$ only 1 of those

$\lvert G\rvert = 12$

$P \in Syl_2$, $Q \in Syl_3$

* Either $Q \trianglelefteq G$ or $G \simeq A_4$

* Abelian: $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$
* Non Ab: A_4, D_{12}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

$\lvert G\rvert = 30$

* $\exists$ $H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$
* Abelian: $\mathbb{Z}_{30}$

$\lvert G\rvert = p^2 q$

$p \neq q$, $P \in Syl_p$, $Q \in Syl_q$

* $p > q \Rightarrow P \trianglelefteq G$
* $q > p$ either $Q \trianglelefteq G$ or $G \simeq A_4$
* Abelian: $\mathbb{Z}_{p^2 q}$, $\mathbb{Z}_{pq} \times \mathbb{Z}_p$

$\lvert G\rvert = 60$

* $n_5 > 1 \Rightarrow G$ simple $\Rightarrow G \simeq A_5$
* Abelian: $\mathbb{Z}_{60}$, $\mathbb{Z}_{30} \times \mathbb{Z}_2$.