Recall

$H, K \trianglelefteq G$

- $H \cap K = 1$

$HK \overset{\sim}{\longleftrightarrow} \{(h,k) \mid h \in H, k \in K\}$

& under this bijection

$(h_1,k_1)\cdot(h_2,k_2) \leftrightarrow (h_1k_1, h_2k_2)$

i.e. $HK \simeq H \times K$

$H, K \leq G$. $H \trianglelefteq G$

$H \cap K = 1$ $\hookrightarrow K \circlearrowright H$

$k * h = khk^{-1}$

$HK \overset{\sim}{\longleftrightarrow} \{(h,k) \mid h \in H, k \in K\}$

$hk \longleftarrow (h,k)$

$(h_1k_1)(h_2k_2) \longleftarrow (h_1,k_1)\cdot(h_2,k_2)$

$= h_1k_1h_2k_1^{-1}k_1k_2$ $\quad (h_1(k_1 * h_2), k_1k_2)$

$\underbrace{(h_1 k_1 * h_2)}_{\in H}\underbrace{(k_1k_2)}_{\in K}$

Multiplication in HK relies on H, K, $K \circlearrowright H$ *by automorphisms

$HK \longleftrightarrow \{(h,k)\}$ but multiplication isn't componentwise.

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*$K \circlearrowright H$ is an action by automorphisms.

$K \xrightarrow{k \mapsto \sigma_k} S_H$

$\searrow \mathrm{Aut}(H) \leq S_H$

$\sigma_k : H \xrightarrow{\simeq} H$

$h \longmapsto khk^{-1}$

Defn H, K be groups.

$\phi : K \longrightarrow \mathrm{Aut}(H)$ a hom.

ϕ defines an action $K \circlearrowright H$

$k * h = \phi(k)(h)$

Let $G = \{(h,k) \mid h \in H, k \in K\}$

define multiplication

$(h_1,k_1)(h_2,k_2)$

$= (h_1\, k_1 * h_2, k_1k_2)$

Then G is called the semidirect product of H & K wrt ϕ. Denoted

$H \rtimes_\phi K$

or

$H \rtimes K$.

Thm H, K, $\phi : K \longrightarrow \mathrm{Aut}H$

$G = H \rtimes_\phi K$

① G is a group &

$|G| = |H| \cdot |K|$.

② $H \xrightarrow{\sim} \{(h,1) \mid h \in H\} \leq G$

$K \xrightarrow{\sim} \{(1,k) \mid k \in K\} \leq G$

Identify $H, K \leq G$.

③ $H \trianglelefteq G$.

④ $H \cap K = 1$

⑤ $h \in H$, $k \in K$. $\leq G$

$khk^{-1} = k \cdot h$

$= \phi(k)(h) \in H$.

Pf/ ① HW 10

$id = (1,1)$

$(h,k)^{-1} = (k^{-1}\cdot h^{-1}, k^{-1})$.

Order part easy

Set $H \rtimes K$ same $H \times K$.

② $\tilde{H} = \{(h,1)\} \leq G$.

$(a,1)(b,1) = (a\, 1\cdot b, 1)$

$= (ab, 1)$.

$H \longrightarrow G$ inj

$h \longmapsto (h,1)$ is a hom

Image is $\tilde{H}$. ✓

$\tilde{K} = \{(1,k)\}$

$(1,c)\cdot(1,d) = (1\cdot c\cdot 1, cd)$

$= (c \cdot 1, cd)$

$= (1, cd)$

B/c $h \mapsto c\cdot h$ is an iso.

$K \longrightarrow G$ inj hom w/ image $\tilde{K}$.

$\tilde{H} \cap \tilde{K} = \{(h,1)\} \cap \{(1,k)\}$

$= \{(1,1)\} = 1$. ④ ✓

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③ & ⑤ Fix $h \in H$, $k \in K$ $\leq G$

$khk^{-1} = (1,k)(h,1)(1,k)^{-1}$

$= (1,k)(h,1)(1,k^{-1})$

$= (k\cdot h, k)(1,k^{-1})$

$= (k\cdot h\ k\cdot 1, kk^{-1})$

$= (k\cdot h, 1)$

$= k\cdot h \in \tilde{H}$

Proves ⑤

Showed $h \in H$, $k \in K$

$khk^{-1} = k\cdot h \in H$

so $K \leq N_G(H)$

& $H \leq N_G(H)$

$HK \leq N_G(H)$

$\|$

G $\quad |HK| = \frac{|H|\cdot|K|}{|H \cap K|} = |G|$

$\Rightarrow H \trianglelefteq G$.

Rmks

* $H \rtimes K$ emphasizes $H \trianglelefteq G$.

* This is not symmetric in general.

$(H \rtimes K \neq K \rtimes H)$

$K \to \mathrm{Aut}H$ $\quad H \to \mathrm{Aut}K$.

* $H \trianglelefteq H \rtimes K$

$K \leq H \rtimes K \rightsquigarrow HK$

$H \cap K = 1$ $\quad \simeq H \rtimes K$.

Prop $H \trianglelefteq G$, $K \leq G$, $H \cap K = 1$

$\phi : K \longrightarrow \mathrm{Aut}H$ conj.

$HK \simeq H \rtimes_\phi K$.

Pf/ $HK \longrightarrow H \rtimes_\phi K$

$hk \longmapsto (h,k)$.

Bijective

& Hom by ✱

H, K ✓	H, K, $\phi : K \to \mathrm{Aut}H$
$H \times K = \{(h,k)\}$	$H \rtimes K = \{(h,k)\}$
$(a,b)(c,d)$	$(a,b)(c,d)$
$= (ac, bd)$	$= (a\ b\cdot c, bd)$.
$H, K \trianglelefteq G$	H, K $H \trianglelefteq G$
$H \cap K = 1$	$H \cap K = 1$
HK	HK

ext (first three rows); internal (last three rows)

Note Always a bijection of sets

$H \times K \xrightarrow{\Phi^{-1}} H \rtimes K$

$(h,k) \longmapsto (h,k)$

Thm Tells us when its an isom.

Thm H, K. $\phi : K \longrightarrow \mathrm{Aut}H$

The following are equivalent

① Φ is a homom ($\Rightarrow$ an isom).

② ϕ is trivial map.

③ $K \trianglelefteq H \rtimes K$.

Pf/

① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ①

Rename $\Phi : H \rtimes K \longrightarrow H \times K$

$(h,k) \longrightarrow (h,k)$

① $\Rightarrow$ ②

$x = (h_1, k_1)$

$y = (h_2, k_2)$ $\quad \in H \rtimes K$

$xy = (h_1\ k_1\cdot h_2, k_1k_2)$

$\Phi(xy)$

$\Phi(x)\Phi(y) = (h_1,k_1)(h_2,k_2)$

$= (h_1h_2, k_1k_2)$.

Assume

$\Phi(xy) = \Phi(x)\Phi(y)$

$h_1\ k_1\cdot h_2 = h_1h_2$ in H

$k_1 \cdot h_2 = h_2$

Varying h's & k's

Shows $k\cdot h = h$ $\forall k, h$

$\Rightarrow K \circlearrowright H$ is trivial

② $\Longrightarrow$ ③

$K \circlearrowright H$ is trivial.

$k \in K$, $h \in H$

$kh = (1,k)(h,1)$

$= (k\cdot h, k)$

$= (h,k)$

$= (h,1)(1,k)$

$= hk$.

$h^{-1}kh = k$

$H \leq N_G(K)$.

So is K.

So $H \rtimes K = HK \leq N_G(K)$

$\Rightarrow K \trianglelefteq H \rtimes K$.

Assuming ③ $\Rightarrow$ ①

$K \trianglelefteq H \rtimes K$

$H, K \trianglelefteq H \rtimes K$ &

$H \cap K = 1$

$\Longrightarrow$ $h \in H$, $k \in K$

$hk = kh$.

$khk^{-1} = k\cdot h$

$\|$

$x = (h_1,k_1)$ $\quad \Phi(x)\cdot\Phi(y)$

$y = (h_2,k_2)$ $\quad = (h_1,k_1)(h_2,k_2)$

$= (h_1h_2, k_1k_2)$

$xy = (h_1\ k_1\cdot h_2, k_1k_2)$

$\Phi(xy) = (h_1h_2, k_1k_2)$

Done

Rmk

*Direct products are semidirect products wrt trivial actions.

* $(a,b)(c,d) = (a\ b\cdot c, bd)$

b gives c a little kick.