**Recall**: $G$ a group, $x,y \in G$

$[x,y] = x^{-1}y^{-1}xy \in G$

$[G,G] = \langle [x,y] \rangle$

**Prop**:
1) $[G,G]$ is the minimal normal $N \trianglelefteq G$ s.t. $G/N$ abelian.
2) $\sigma \in \mathrm{Aut}\, G$

$\sigma[x,y] = [\sigma(x), \sigma(y)]$

**Examples**
1) $G$ abelian $\Leftrightarrow [G,G] = 1$
2) $[Q_8, Q_8]$
   1) $Q_8/\langle -1\rangle \cong V_4$ abelian

$\Rightarrow 1 \le [Q_8,Q_8] \le \langle -1\rangle$

$\Rightarrow [Q_8,Q_8] = \cancel{1}$ or $\langle -1\rangle$

But $Q_8/1$ not abelian.

3) $[D_{2n}, D_{2n}] = \langle r^2 \rangle$

Pf ① $[r,s] = r^{-1}s^{-1}rs = r^{-2} \in [D_{2n}, D_{2n}]$

$\Rightarrow \langle r^2\rangle \le [D_{2n}, D_{2n}]$.

$G = D_{2n}/\langle r^2\rangle = \langle \bar r, \bar s\rangle$ but both have order 2, which commute ($\bar r\bar s = \bar s\bar r^{-1} = \bar s \bar r$)

$\Rightarrow G$ abelian.

$\Rightarrow \langle r^2\rangle \ge [D_{2n}, D_{2n}]$

∎

---

$[S_n, S_n] \ni \sigma = [x,y]$

Then also contains $\tau\sigma\tau^{-1}$ $\forall \tau \in S_n$

$\tau\sigma\tau^{-1} = \tau[x,y]\tau^{-1} = [\tau x\tau^{-1}, \tau y \tau^{-1}] \in [S_n, S_n]$.

$\Rightarrow \forall$ things w/ same cycle type as $\sigma$ are in $[S_n, S_n]$.

e.g. $S_5 \ge D_{10} \ni r, s$

$r = (1\ 2\ 3\ 4\ 5)$

$s = (2\ 5)(3\ 4)$

$[r,s] = r^{-2} = (1\ 4\ 2\ 5\ 3)$

So says $(1\ 4\ 2\ 3\ 5) \in [S_5, S_5]$

$\Rightarrow \forall$ 5-cycles in $[S_5, S_5]$.

---

Recall from right before proving isom thms

**Prop** $H, K \le G$

$$|HK| = \frac{|H|\cdot|K|}{|H\cap K|}$$

How many ways can we write $hk \in H\cdot K$ w/ $h \in H$, $k \in K$

Claim $|H \cap K|$ many ways

{ways to write $hk$} $\longleftrightarrow H\cap K$.

$hk = \underbrace{hx}_{\in H}\underbrace{x^{-1}k}_{\in K} \longleftarrow x \in H\cap K$

If $hk = h'k'$

$\underbrace{(h')^{-1}h}_{H} = \underbrace{k'k^{-1}}_{K} = x \in H\cap K$

---

**Prop**: $H, K \le G$. $x \in HK$

① # of ways to write $x = hk$ w/ $h \in H$ & $k \in K$ is $|H\cap K|$

② If $H \cap K = 1$ Then $x$ can be written uniquely as a product $hk$ w/ $h \in H$ & $k \in K$

---

This allows us to recognize direct products.

**Thm** Suppose $H, K \le G$.

Assume
① $H, K \trianglelefteq G$.
② $H \cap K = 1$

Then $HK \cong H \times K$.

**Proof**

Notice 2nd iso $\Rightarrow HK \le G$

Claim $h \in H$, $k \in K \Rightarrow hk = kh$.

Pf/ $H \trianglelefteq G \Rightarrow k^{-1}hk \in H$

$\Rightarrow h^{-1}k^{-1}hk \in H$

$= [h,k]$

$K \trianglelefteq G \Rightarrow h^{-1}k^{-1}h \in K$

$(h^{-1}k^{-1}h)k \in K$

$= [h,k]$

$\Rightarrow [h,k] \in H \cap K$

$= 1$

$\Rightarrow hk = kh$ ∎

Define a map

$HK \xrightarrow{\phi} H \times K$

$x = hk \longmapsto (h,k)$

Uniquely by prop

so $\phi$ is well defined.

---

**Hom**: $x_1 = h_1k_1$, $x_2 = h_2k_2 \in HK$

$\phi(x_1x_2) = \phi(h_1k_1h_2k_2)$

$= \phi(h_1h_2 \cdot k_1k_2)$

$= (h_1h_2, k_1k_2)$

$= (h_1,k_1)(h_2,k_2)$

$= \phi(h_1k_1)\phi(h_2k_2)$

$= \phi(x_1)\phi(x_2)$.

**Surj** $(h,k) = \phi(hk)$

**Inj** $\phi(hk) = (1,1) = (h,k)$

$h=1$, $k=1 \Rightarrow hk = 1$ ∎

---

**Defn/Notation** $H,K \trianglelefteq G$, $H\cap K = 1$

$H \times K \leftarrow$ external direct product. (not depend on G)

S/

$HK \leftarrow$ internal direct product

**Ex** $\mathbb{Z}_{nm}$ $(n,m) = 1$

$|x| = n$ $|y| = m$

$\langle x\rangle = H \cong \mathbb{Z}_n$

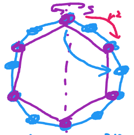
$\langle y \rangle = K \cong \mathbb{Z}_m$

$\mathbb{Z}_n\mathbb{Z}_m \ni x^iy^j \in \mathbb{Z}_{nm}$

$\mathbb{Z}_n \times \mathbb{Z}_m \ni (x^i, y^j)$

**Ex** If $n$ is odd

$\Rightarrow D_{4n} \cong D_{2n} \times \mathbb{Z}_2$



12-gon

$r, s$ restricts to a sym of 6-gon

$D_{4n} = \langle r, s \mid r^{2n} = s^2 = 1, rs = sr^{-1}\rangle$

---

$\langle s, r^2 \rangle = H \cong D_{2n}$

$\langle r^n \rangle = K \cong \mathbb{Z}_2$

$H \trianglelefteq D_{4n}$ index 2.

$sr^ns^{-1} = r^{-n} = r^n$

$\Rightarrow K \trianglelefteq D_{4n}$.

$r^n \notin \langle s, r^2\rangle$

So $K \cap H = 1$.

So Thm

$HK \cong H \times K$

$H \lneq HK \le D_{4n}$

$2n \quad 4n \quad 4n$

$HK = D_{4n} = H \times K$

$\cong D_{2n} \times \mathbb{Z}_2$. ∎

---

**Recognize a direct product**

$H, K \trianglelefteq G$, $H \cap K = 1$ $\Rightarrow HK \cong H \times K$ $\Rightarrow HK/H \cong K$

① $x \in HK$ $x = hk$ uniquely

② $hk = kh$. ③ $HK$ a gp.

**Upshot** ↑

Algebra of $HK \le G$ no longer depends on $G$ just on $H, K$.

What if we weaken our assumptions??

---

Let $H, K \le G$

① $H \trianglelefteq G$

② $H \cap K = 1$.

---

**Do some computing**

Understand $HK \le G$

$x \in HK$ since $H \cap K = 1$

$\Rightarrow x = hk$ uniquely w/ $h \in H$ & $k \in K$.

$x_1 = h_1k_1$ & $x_2 = h_2k_2$

$x_1x_2 = h_1k_1h_2k_2$ | as $hk$

$= h_1k_1h_2(k_1^{-1}k_1)k_2$

$= \underbrace{h_1(k_1h_2k_1^{-1})}_{h \in H} \underbrace{k_1k_2}_{k \in K}$.

**Therefore** $K \circlearrowright H$ via conjugation

$hk \in HK \longleftrightarrow (h,k)$ $h \in H$, $k \in K$

$(h_1,k_1)\cdot(h_2,k_2)$

$= (h_1(k_1 \cdot h_2), k_1k_2)$

**Upshot**

Multiplication in $HK$ depends on $H$, $K$, $K \circlearrowright H$

intrinsic to $H$ & $K$.