2 perspectives ab gps size n

Inv Factors	Elementary divisors
$(n_1, n_2, \dots, n_s)$	$\forall\, p \mid n$ max p-div $(\beta_1, \dots, \beta_n) \neq 1$
① $n_i \geq 2$	① $\beta_i \geq 1$
② $n_{i+1} \mid n_i$	② $\beta_{i+1} \geq \beta_i$
③ $n_1 \cdots n_s = n$	③ $\sum \beta_i = \alpha$
$\Downarrow$ $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_s}$	$n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$; $A_i = \mathbb{Z}_{p_i^{\beta_{i1}}} \times \dots \times \mathbb{Z}_{p_i^{\beta_{it}}}$; $A_1 \times A_2 \times \dots \times A_t$
$\mathbb{Z}_{90} \times \mathbb{Z}_2$ $(90, 2)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_{3^2}) \times \mathbb{Z}_5$; $p=2$: $\beta_1 = \beta_2 = 1$; $p=3$: $\beta_1 = 2$; $p=5$: $\beta_1 = 1$

Inv Factors ⇒ Elem divs

Main tool $n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$

$$\mathbb{Z}_n = \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_n^{\alpha_n}}.$$

Example

* $\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2$ $//A$ $(30,30,2)$

$30 = 2 \cdot 3 \cdot 5$

$\mathbb{Z}_{30} = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_{30} = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 = \mathbb{Z}_2$

$\Rightarrow A \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5)$

Rmk This only needed decomposition into a product of cyclic gps (inv. factors not impt).

Example

$\mathbb{Z}_{15} \times \mathbb{Z}_{20}$, $15 = 3 \cdot 5$, $20 = 2^2 \cdot 5$ $\simeq (\mathbb{Z}_{2^2}) \times (\mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5)$

$\mathbb{Z}_{15} = \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{20} = \mathbb{Z}_4 \times \mathbb{Z}_5$

Prop If G is a product of cyclic groups, it has a unique elementary divisor decomp.

Rmk Thus to prove fund thm of fin ab. gps. suffices to show any ab. gp is a product of cyclic groups

Elem Divs ⟹ inv factors

Setup G abelian gp of order $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$

Have elem div decomp.

Step 1: Group elem divisors which are power of the same prime together (into t lists).

Ex $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \times \mathbb{Z}_5 \simeq \mathbb{Z}_{90} \times \mathbb{Z}_2$

p=2	p=3	p=5	
2	3	5	$n_1 = 2 \cdot 3 \cdot 5 = 90$
2	1	1	$n_2 = 2 \cdot 1 \cdot 1 = 2$.

Step 2 rearrange into non increasing order.

Step 3 Say longest list has k elements. Fill out the rest of the lists w/ 1s until they all have length k.

Step 4 For each $i = 1$ to k make n_i the product of the ith elt of each list.

Example $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5)$

p=2	p=3	p=5
2	3	5
2	3	5
2	1	1

$n_1 = 2 \cdot 3 \cdot 5 = 30$

$n_2 = 2 \cdot 3 \cdot 5 = 30$

$n_3 = 2 \cdot 1 \cdot 1 = 2$

$\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2$

$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25})$

p=2	p=3	p=5	n_i
4	3	25	300
2	3	1	6
1	3	1	3

5×50

$\mathbb{Z}_{300} \times \mathbb{Z}_6 \times \mathbb{Z}_3$

Remark Uniqueness part of fundamental theorem lets us easily check if abelian gps are isomorphic.

Example

$6 = 2 \cdot 3$, $15 = 3 \cdot 5$

$\mathbb{Z}_6 \times \mathbb{Z}_{15} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\not\simeq$

$\mathbb{Z}_{10} \times \mathbb{Z}_9 \simeq \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$10 = 2 \cdot 5$, $9 = 3^2$

Defn $G \simeq \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_t}$ Inv Factor Decomp

$r :=$ free rank

$t :=$ torsion rk

$G \simeq \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_t}$

$t :=$ rank

Defn Exponent of G

$\mathrm{Exp}(G) = \min\{n \mid x^n = 1 \ \forall x \in G\}$

Rmk $G \simeq \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_t}$ I.F.D

$\mathrm{Exp}(G) = n_1$.

Commutators:

Defn G a group

① $x, y \in G$ the commutator of x & y is $[x,y] = x^{-1}y^{-1}xy \in G$

② $A, B \leq G$ then $[A,B] = \langle [x,y] \mid x \in A, y \in B \rangle$

③ $[G,G]$ is the commutator subgroup of G.

Noticed in HW that $[x,y] = 1 \iff xy = yx$.

Prop G a group, $x, y \in G$, $H \leq G$

① $xy = yx[x,y]$

② $H \trianglelefteq G \iff [H,G] \leq H$

③ $\sigma \in \mathrm{Aut}\, G$. $\Rightarrow \sigma[x,y] = [\sigma(x), \sigma(y)]$

④ $[G,G]$ char G. in particular $[G,G] \trianglelefteq G$.

⑤ $G/[G,G]$ is abelian. ← called G^{ab}

⑥ $H \trianglelefteq G$. Then G/H abelian $\iff [G,G] \leq H$.

⑦
$$\begin{tikzcd} G \arrow[r] \arrow[d] & A \\ G/[G,G] \arrow[ur, dotted] & \end{tikzcd}$$
A ← abelian; exist unique

pf/ ① Immediate

$yx[x,y] = yxx^{-1}y^{-1}xy$
$= yy^{-1}xy$
$= xy$ ✓

② $H \trianglelefteq G \iff g^{-1}hg \in H$ $\forall h \in H$, $g \in G$

$\iff h^{-1}g^{-1}hg \in H$

$\iff [h,g] \in H$ $\forall h \in H$, $g \in H$

$\iff [H,G] \leq H$.

③ $\sigma \in \mathrm{Aut}(G)$

$\sigma([x,y]) = \sigma(x^{-1}y^{-1}xy)$
$= \sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y)$
$= [\sigma(x), \sigma(y)]$ ✓

④ Recall H char G: $\forall \sigma \in \mathrm{Aut}\, G$ $\sigma(H) = H$.

$\sigma \in \mathrm{Aut}\, G$

$\sigma([G,G]) \leq [G,G]$

Do for σ^{-1} as well ✓

⑤ HW6 prob 6

⑥ G/H ab $\iff xHyH = yHxH$ $\forall x, y \in G$

$\iff y^{-1}x^{-1}yxH = H$

$\iff [x,y] \in H$

$\iff [G,G] \leq H$

⑦ HW 6