Projective Geometry for Perfectoid Spaces

Gabriel Dorfsman-Hopkins

University of Washington

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Outline

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How precise can we make this?

The Fontaine-Wintenberger Isomorphism

Theorem (Fontaine-Winterberger)

There is a canonical isomorphism of absolute Galois groups

$$\mathrm{Gal}\left(\mathbb{Q}_p\left(p^{1/p^\infty}\right)\right) \cong \mathrm{Gal}\left(\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)\right).$$

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Question

Is this a manifestation of a geometric correspondence on the level of points?

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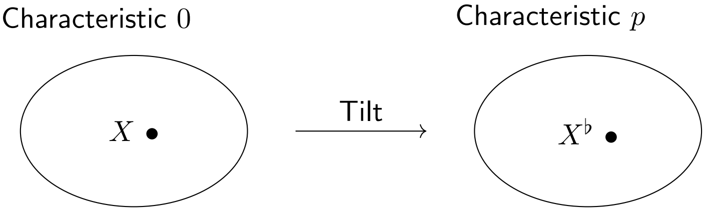

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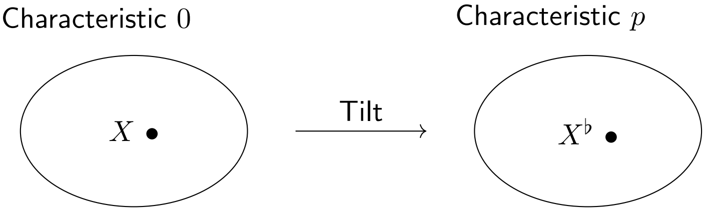

This is an equivalence!

The Tilting Equivalence

Theorem (Scholze)

Let S be a perfectoid space with tilt $S^\flat$. The functor $X \mapsto X^\flat$ is an equivalence of categories from perfectoid spaces over S to perfectoid spaces over $S^\flat$, inducing an equivalence of étale sites:

$$S_{ét} \xrightarrow{\sim} S^\flat_{ét}.$$

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Letting S be the perfectoid space associated to $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$, then $S^\flat$ is the perfectoid space associated to $\mathbb{F}_p((t^{1/p^\infty}))$, and so we recover the Fontaine-Wintenberger isomorphism:

$$\mathrm{Gal}\left(\mathbb{Q}_p\left(p^{1/p^\infty}\right)\right) \cong \mathrm{Gal}\left(\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)\right).$$

Goal

Question

Can we develop a reasonable notion of projective geometry for perfectoid spaces?

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Analogy to Algebraic Geometry

Algebraic Geometry	Perfectoid Geometry

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Rings		

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Remark

Let $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ be the pth power map on coordinates. Then:

$$\mathbb{P}^{n,perf} \sim \varprojlim \left(\cdots \xrightarrow{\varphi} \mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^n \right).$$

Compatibility

Lemma (Scholze)

Let K be a perfectoid field with tilt $K^\flat$.

$$\begin{aligned}\left(\mathbb{D}_K^{n,perf}\right)^\flat &\cong \mathbb{D}_{K^\flat}^{n,perf}\\ \left(\mathbb{P}_K^{n,perf}\right)^\flat &\cong \mathbb{P}_{K^\flat}^{n,perf}.\end{aligned}$$

Line Bundles on The Disk

Theorem (D-H,Kedlaya)

Finite vector bundles on $\mathbb{D}^{n,perf}$ *are all trivial.*

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The analogous statement for $\mathbb{A}^n$ is known as the Quillen-Suslin theorem, and was proven in 1976.

Line Bundles on Projectivoid Space

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	$\mathbb{P}^n$	$\mathbb{P}^{n,perf}$
Picard Group	$\mathbb{Z}$	$\mathbb{Z}[1/p]$
$\mathcal{O}(d)$	Homogeneous polynomials of degree d in $k[x_0, \cdots, x_n]$	Homogenous power series of degree d in $K\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\rangle$

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Maps to Projectivoid Space

Like in classical geometry, maps to projectivoid space can be expressed in terms of globally generated line bundles.

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Theorem (D-H)

Let X be a perfectoid space over K. A map $X \to \mathbb{P}^{n,perf}$ is equivalent to a sequence of globally generated line bundles $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \cdots)$ on X such that $\mathcal{L}_{i+1}^{\otimes p} \cong \mathcal{L}_i$, together with global sections $s_{i,0}, \cdots, s_{i,n} \in \Gamma(X, \mathcal{L}_i)$ for each i which generate $\mathcal{L}_i$, such that $s_{i+1,j}^{\otimes p} = s_{i,j}$.

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If a map $\varphi : X \to \mathbb{P}^{n,perf}$ is given by this data then: $\varphi^* \mathcal{O}(1/p^i) \cong \mathcal{L}_i$ and $\varphi^*(T_j^{1/p^i}) = s_{i,j}$.

Corollaries

If K has characteristic p, X is perfect, so the pth power map on $\operatorname{Pic} X$ is an isomorphism. Therefore we can refine the theorem.

Corollary

Let X be a perfectoid space over K of positive characteristic. A map $X \to \mathbb{P}^{n,perf}$ is equivalent to a line bundle on X together $n+1$ generating global sections.

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Corollary

Let X be a perfectoid space over K of positive characteristic. A map $X \to \mathbb{P}^{n,perf}$ is equivalent to a line bundle on X together $n+1$ generating global sections.

The tilting equivalence simplifies matters further. Since $\operatorname{Hom}\left(X, \mathbb{P}_K^{n,perf}\right) = \operatorname{Hom}\left(X^\flat, \mathbb{P}_{K^\flat}^{n,perf}\right)$, we have:

Corollary

Let X be a perfectoid space over K of any characteristic. A map $X \to \mathbb{P}_K^{n,perf}$ is equivalent to a line bundle on $X^\flat$ together with $n+1$ generating global sections.

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We can use this compare the Picard groups of a perfectoid space and its tilt.

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Setup

If X is a perfectoid space, $X^\flat$ is homeomorphic, so we can view their multiplicative group sheaves $\mathbb{G}_m$ and $\mathbb{G}_m^\flat$ as sheaves on the same topological space. In fact,

$$\mathbb{G}_m^\flat \xrightarrow{\cong} \varprojlim_{x \mapsto x^p} \mathbb{G}_m$$

Taking cohomology we get a sequence of maps

$$\operatorname{Pic} X^\flat \longrightarrow \varprojlim_{\mathcal{L} \mapsto \mathcal{L}^p} \operatorname{Pic} X \longrightarrow \operatorname{Pic} X.$$

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Let's use our theorem to study $\operatorname{Pic} X^\flat \to \varprojlim \operatorname{Pic} X$.

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Since $X^\flat$ has characteristic p this corresponds to a map

$$X^\flat \to \mathbb{P}^{n,perf}_{K^\flat}.$$

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The tilting equivalence implies that this corresponds to a unique map

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The main theorem associates to this map a unique sequence

$$(\mathcal{L}_1, \mathcal{L}_2, \cdots) \in \varprojlim \operatorname{Pic} X.$$

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Theorem (D-H)

Suppose X is a perfectoid space over K. Suppose that X has an ample line bundle and that $H^0(X, \mathcal{O}_X) = K$. Then

$$\operatorname{Pic} X^\flat \hookrightarrow \varprojlim_{\mathcal{L} \mapsto \mathcal{L}^p} \operatorname{Pic} X.$$

In particular, if $\operatorname{Pic} X$ has no p torsion, then

$$\operatorname{Pic} X^\flat \hookrightarrow \operatorname{Pic} X.$$

Idea of Proof

Let's consider the case where $\mathcal{L}, \mathcal{M} \in \operatorname{Pic} X^\flat$ are globally generated, and both have the same image. Then choosing sections gives two maps $\phi^\flat$ and $\psi^\flat$ from $X^\flat$ to projectivoid space over $K^\flat$.

Untilt these two maps to ϕ and ψ from X to projectivoid space over K. Combining the sections giving ϕ and those giving ψ gives us the following diagram, which we can then tilt.

$$\begin{tikzcd}
& \mathbb{P}_K^{n,perf} & & & \mathbb{P}_{K^\flat}^{n,perf} \\
X \arrow[ur, "\phi"] \arrow[r, "\gamma"] \arrow[dr, "\psi"'] & \mathbb{P}_K^{n+r+1,perf} \arrow[u, dashed] \arrow[d, dashed] & \xrightarrow{tilt} & X^\flat \arrow[ur, "\phi^\flat"] \arrow[r, "\gamma^\flat"] \arrow[dr, "\psi^\flat"'] & \mathbb{P}_{K^\flat}^{n+r+1,perf} \arrow[u, dashed] \arrow[d, dashed] \\
& \mathbb{P}_K^{r,perf} & & & \mathbb{P}_{K^\flat}^{r,perf}.
\end{tikzcd}$$

Thus $\mathcal{L} = \phi^{\flat *}\mathcal{O}(1) = \gamma^{\flat *}\mathcal{O}(1) = \psi^{\flat *}\mathcal{O}(1) = \mathcal{M}$.

Thank You!